Hydrostatic modelling

- **What** is the hydrostatic approximation ? When is it **valid** ?
- What are the **degrees of freedom** of a hydrostatic model ?
- How do we **prognose** the degrees of freedom and **diagnose** other quantities ?

answer depends on the choice of *vertical coordinate*

Hydrostatic approximation :
basic idea and order-of-magnitude arguments

Vertical momentum balance :

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

If vertical acceleration is small :

$$\frac{\mathrm{d}w}{\mathrm{d}t} \ll g,\ \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \frac{\partial p}{\partial z} = -\rho g$$

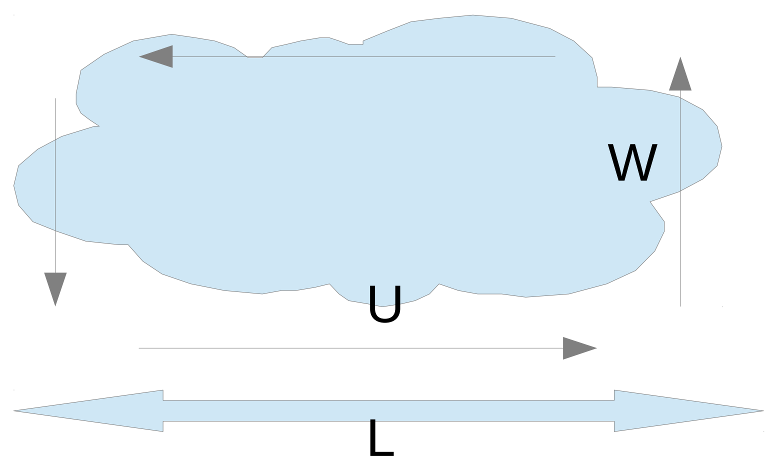
How good is this approximation ?
what we neglect << what is left ?

$$\frac{\mathrm{d}}{\mathrm{d}t} \sim \frac{W}{H} \sim \frac{U}{L} \qquad \frac{\mathrm{d}w}{\mathrm{d}t} \sim \frac{W^2}{H}$$

$$\frac{\partial p}{\partial x} \sim \rho\frac{\mathrm{d}u}{\mathrm{d}t} \sim \rho\frac{U}{L}U$$

$$\frac{\partial p}{\partial z} \sim \frac{L}{H}\frac{\partial p}{\partial x} \sim \frac{\rho U^2}{H}$$

$$\rho\frac{Dw}{Dt}\Big/\frac{\partial p}{\partial z} \sim \left(\frac{W}{U}\right)^2 \sim \left(\frac{H}{L}\right)^2$$



Hydrostatic approximation valid if

$$\frac{H}{L} \ll 1$$

Atmosphere : OK for large-horizontal-scale circulation (>30km)
KO for convection (storms), orographic flow (mountains).

Ocean : H~1km => OK down to kilometer-scale

Hydrostatic approximation from least action principle

$$\frac{D\mathbf{m}}{Dt} - \frac{\partial L}{\partial \mathbf{x}} - \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) = 0 \qquad \mathbf{m} = \frac{\partial L}{\partial \mathbf{u}} \qquad PV = \frac{1}{\rho}\nabla s \cdot (\nabla \times \mathbf{m})$$

$$L = \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \mathbf{R}\cdot\mathbf{u} - \Phi - h(p,s) + \frac{p}{\rho}$$

$$\frac{D\mathbf{u}}{Dt} + \mathrm{curl}\mathbf{R}\times\mathbf{u} + \nabla\Phi + \frac{1}{\rho}\nabla p = 0$$

$$E = \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \Phi + h(p,s) - \frac{p}{\rho} \qquad \mathbf{m} = \mathbf{u} + \mathbf{R}$$

Horizontal projection $\qquad \mathbf{u}_H = \mathbf{u} - u_z\mathbf{e}_z$

$$L = \frac{\mathbf{u}_H\cdot\mathbf{u}_H}{2} + \mathbf{R}\cdot\mathbf{u}_H - \Phi - h(p,s) + \frac{p}{\rho}$$

$$\frac{D\mathbf{u}_H}{Dt} + \mathrm{curl}\mathbf{R}\times\mathbf{u}_H + \nabla\Phi + \frac{1}{\rho}\nabla p = 0$$

$$E = \frac{\mathbf{u}_H\cdot\mathbf{u}_H}{2} + \Phi + h(p,s) - \frac{p}{\rho} \qquad \mathbf{m} = \mathbf{u}_H + \mathbf{R}$$

Prognosing hydrostatic + Boussinesq

$$u, v, s, q$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0$$

$$\frac{1}{\rho_0} \frac{\partial p'}{\partial z} = g b(z, s, q)$$

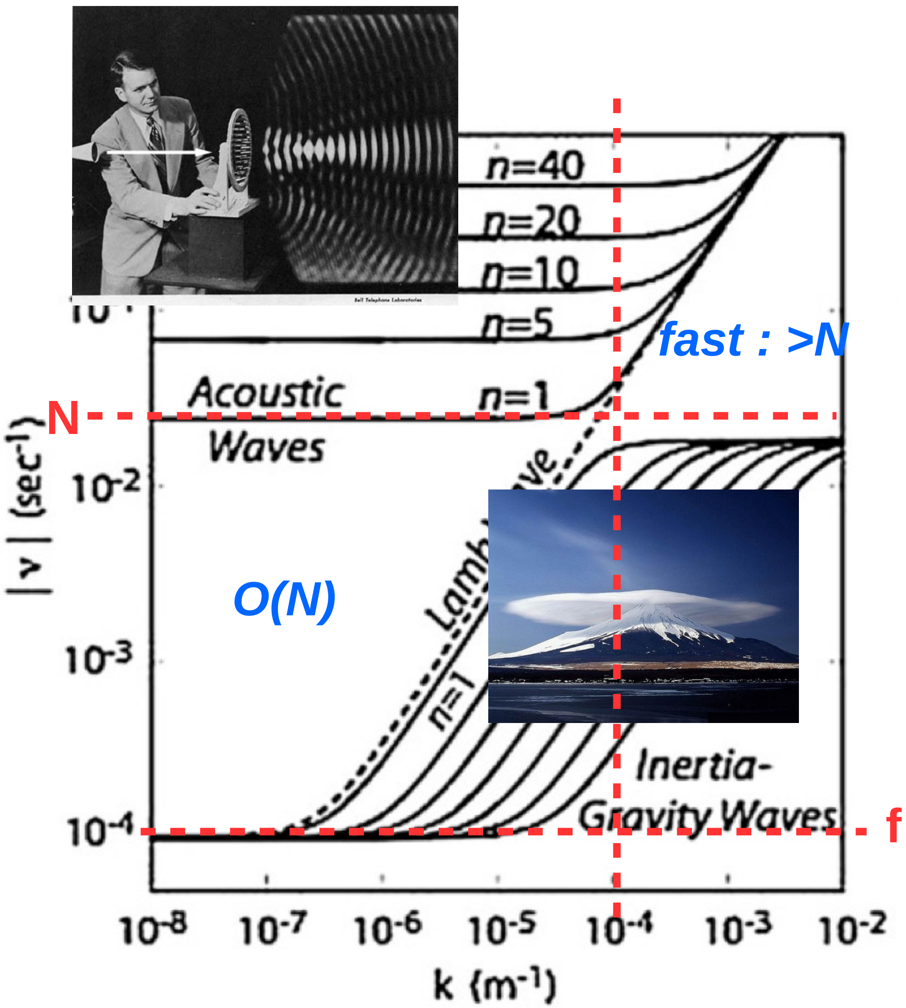
$$\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s = 0$$

$$\frac{\partial \mathbf{u}_H}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}_H + f \mathbf{e}_z \times \mathbf{u}_H + \frac{1}{\rho_0} \nabla_H p' = 0$$

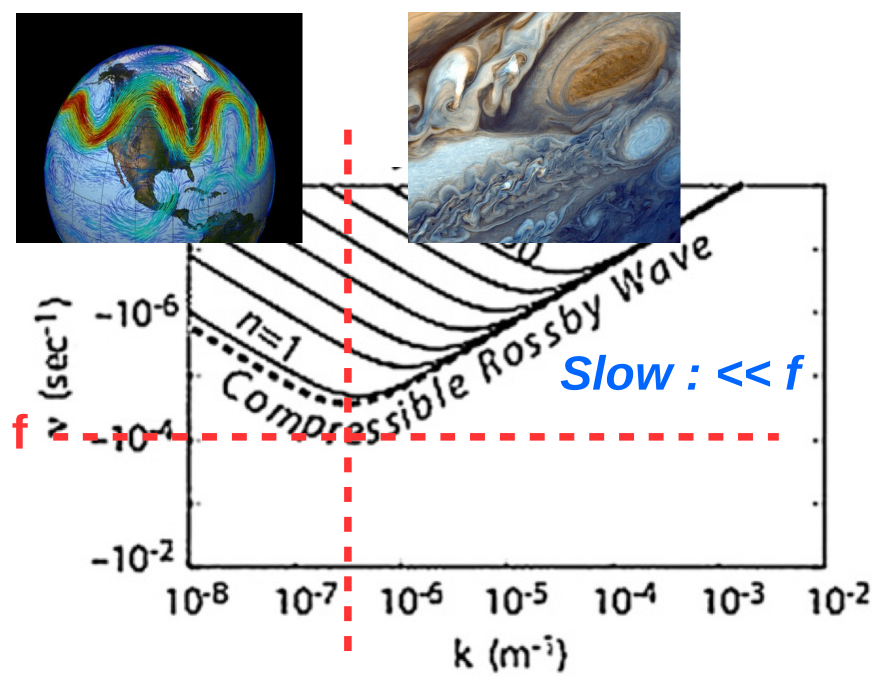
$$\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial s}{\partial t}, \frac{\partial q}{\partial t}$$

Classes of motion in a gravity-dominated, rotating, compressible flow :

Waves in an isothermal atmosphere at rest



f-plane

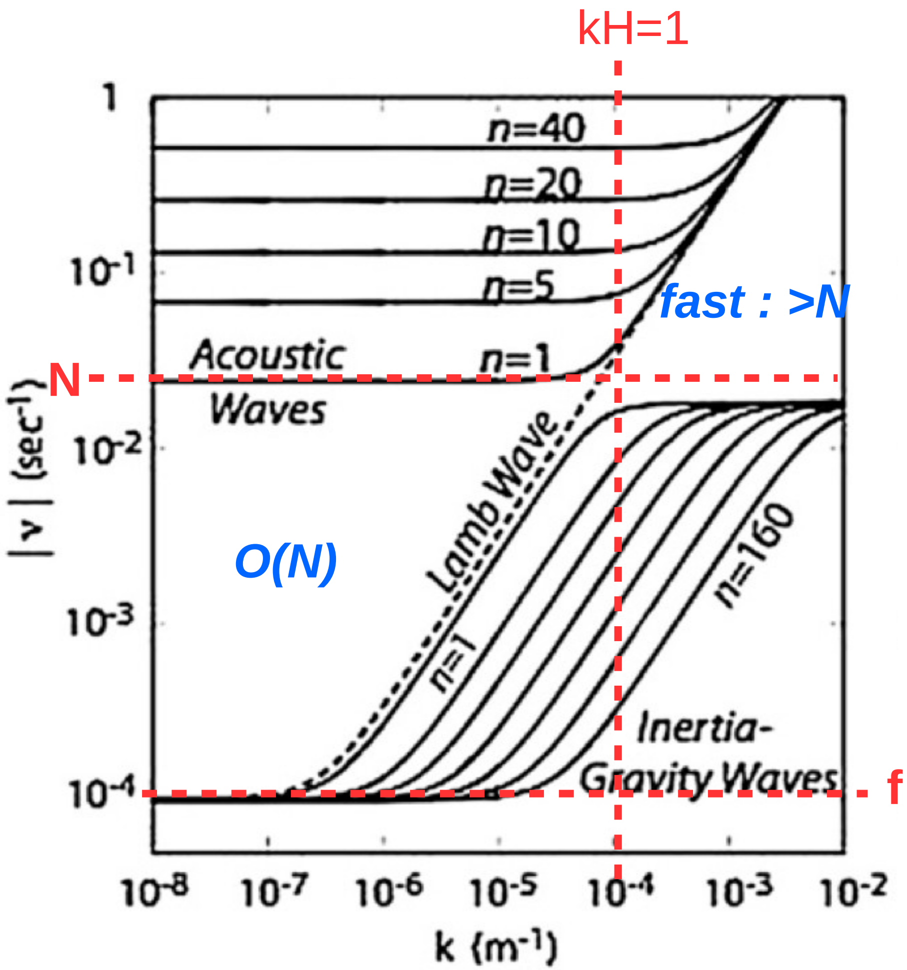


beta-plane

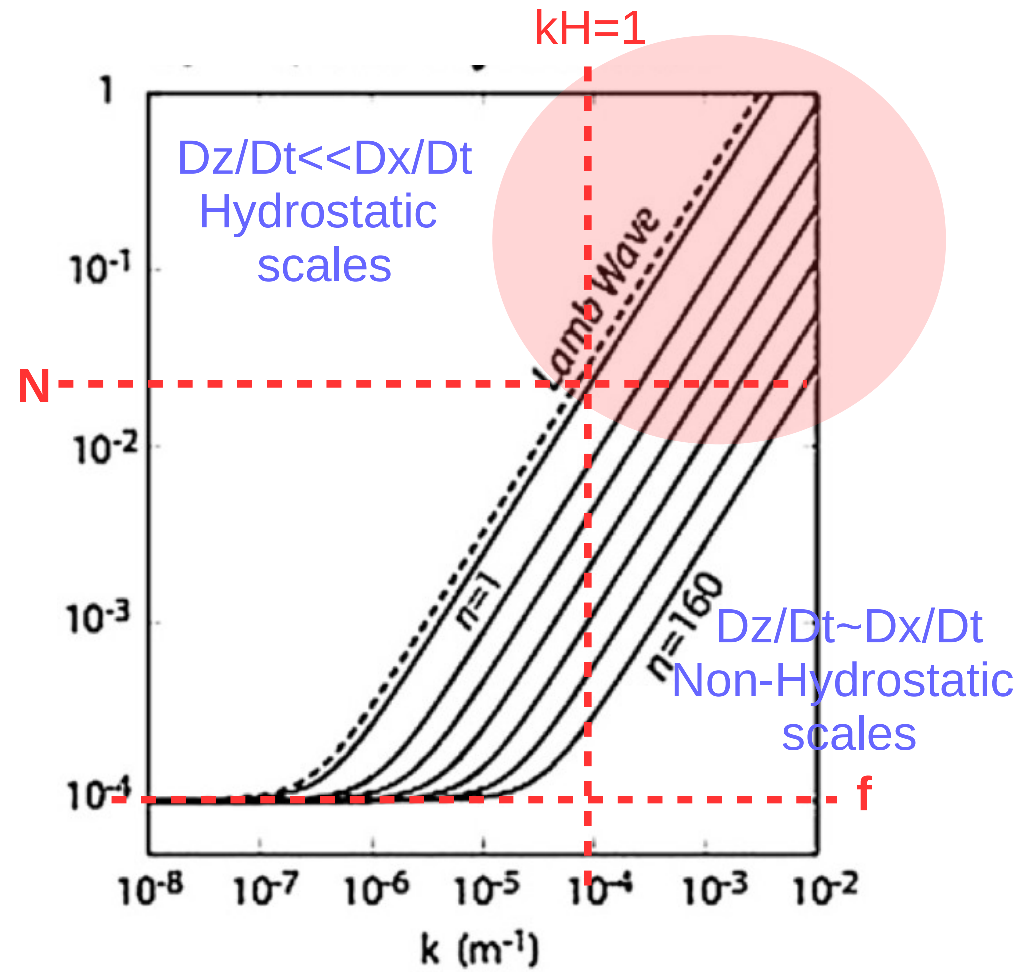
Waves in an isothermal atmosphere at rest

Traditional f-plane approximation

« Exact »



Hydrostatic



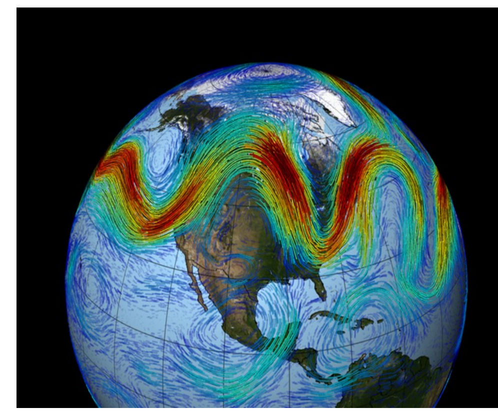
$$\frac{\mathrm{d}w}{\mathrm{d}t}/\rho\frac{\partial p}{\partial z} \sim \left(\frac{W}{U}\right)^2 \sim \left(\frac{H}{L}\right)^2$$

Characteristic scales

- *Velocity* : Sound $c \sim 340$ m/s — Wind $U \sim 30$ m/s
- *Time* : Buoyancy oscillations $N \sim g/c \sim 10^{-2}\,s^{-1}$ — Coriolis $f \sim 10^{-4}\,s^{-1}$
- *Length* : **Scale height $H=c^2/g=10$km** — **Rossby radius : $R=c/f \sim 1000$ km**

Mach number : $M=U/c \ll 1$

Scale separation : $f/N \sim H/R \ll 1$



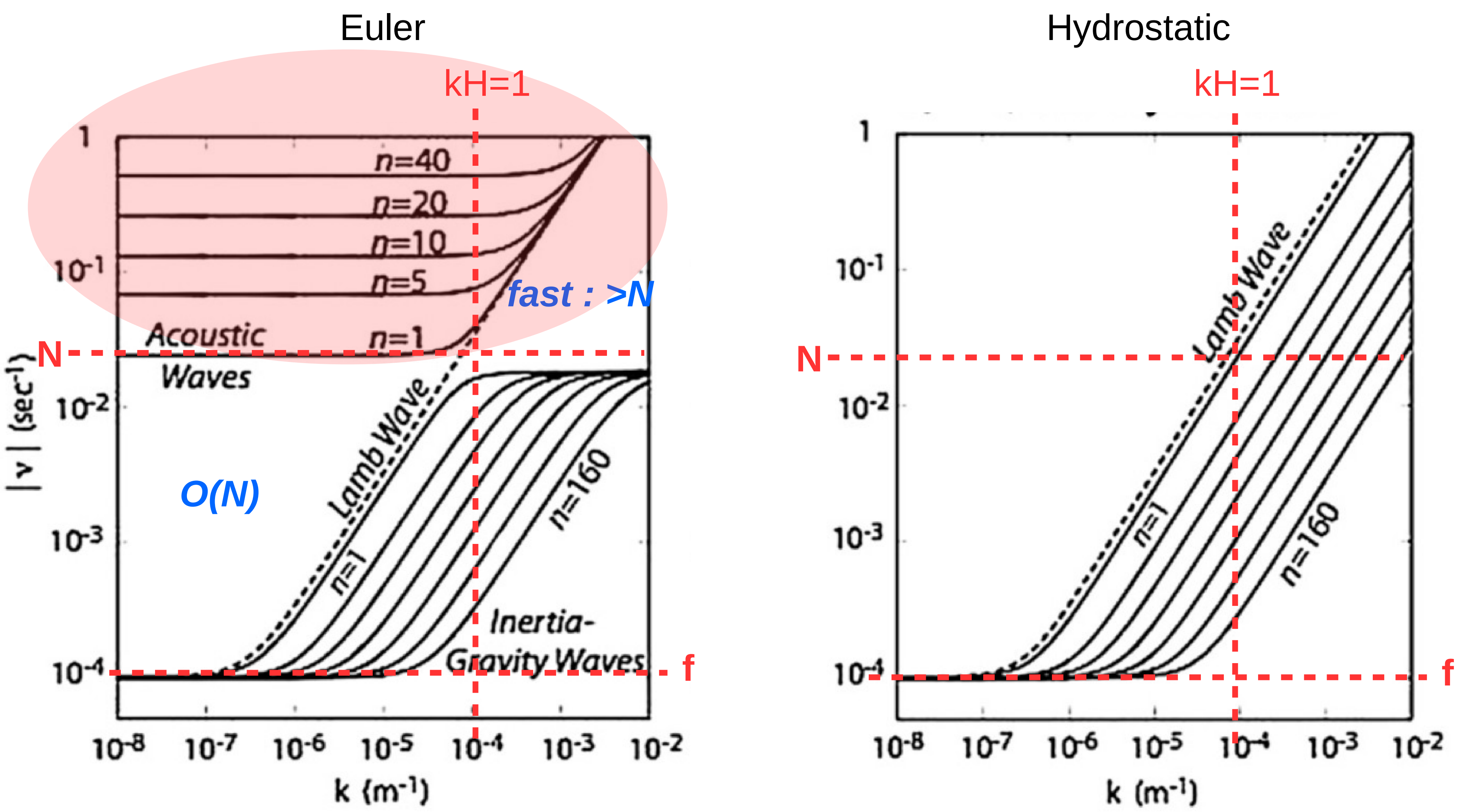
small-scale | scale height | mesoscale | synoptic | planetary

1 km | 10 km | 100 km | 1000 km | 10000 km

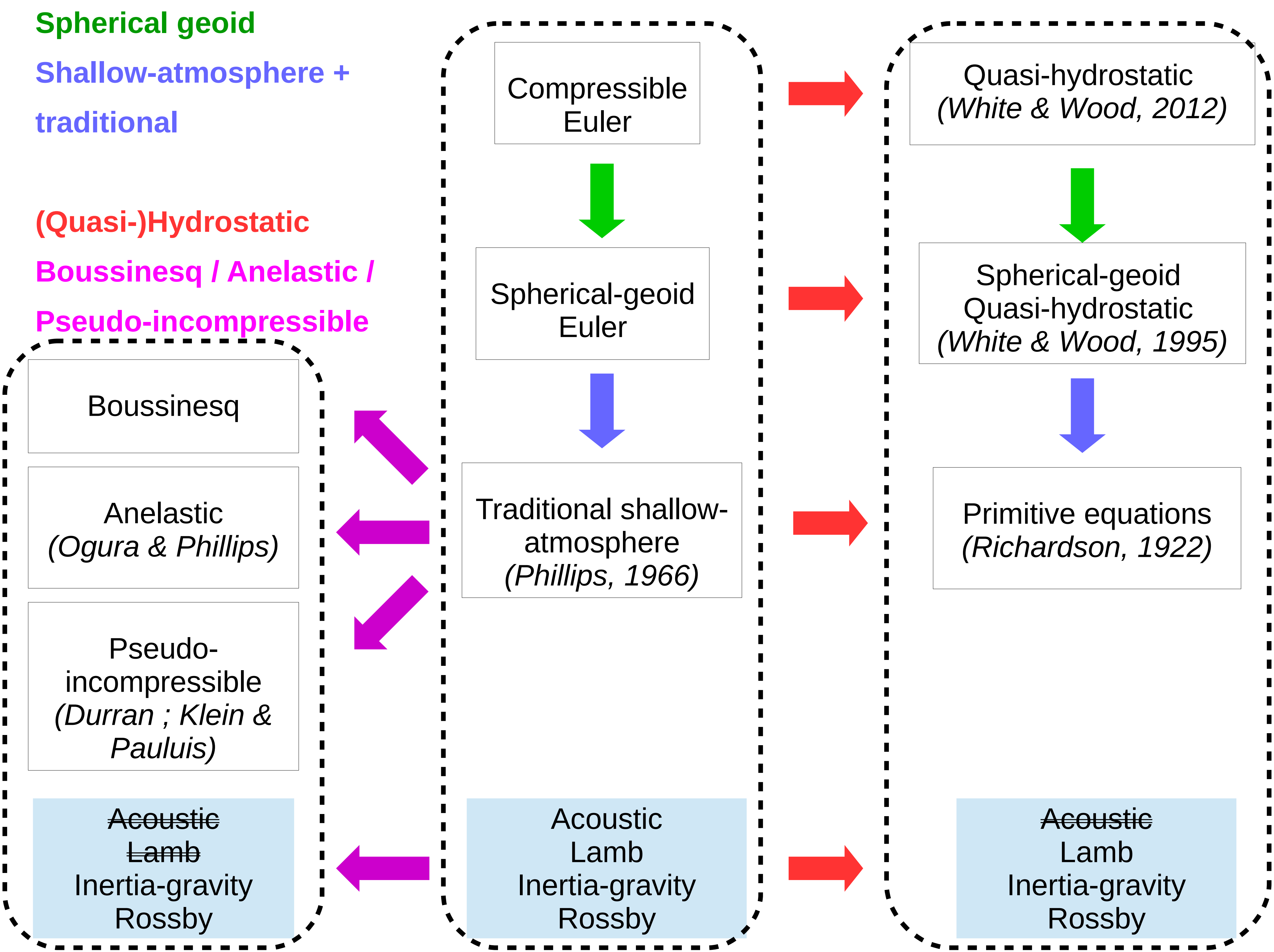
non-hydrostatic | **hydrostatic**

Waves in an isothermal atmosphere at rest

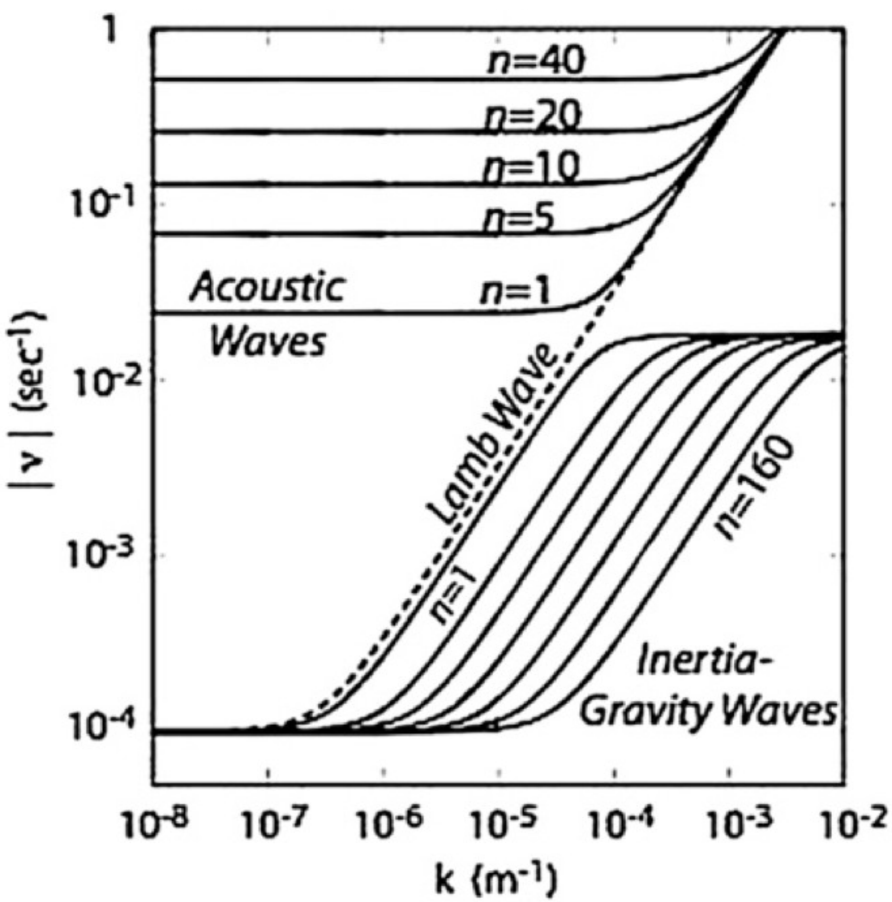
Traditional f-plane approximation

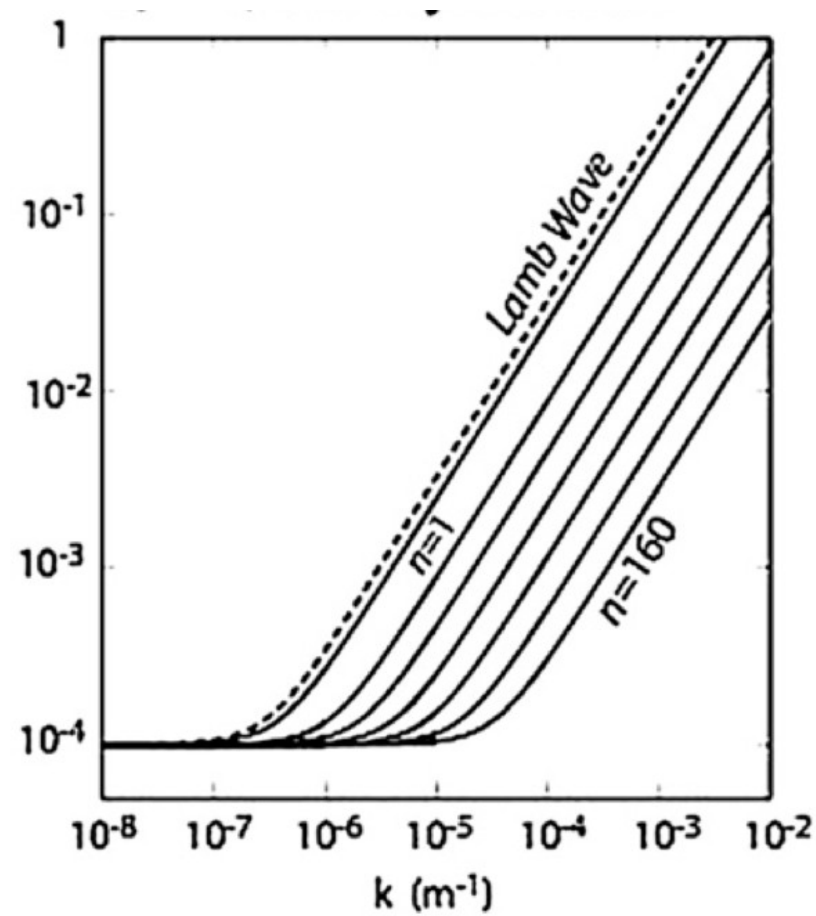


Acoustic waves suppressed : « sound-proof » approximation


Spherical geoid
Shallow-atmosphere + traditional
(Quasi-)Hydrostatic
Boussinesq / Anelastic / Pseudo-incompressible
Compressible Euler
Spherical-geoid Euler
Traditional shallow-atmosphere (Phillips, 1966)
Acoustic
Lamb
Inertia-gravity
Rossby
Quasi-hydrostatic (White & Wood, 2012)
Spherical-geoid Quasi-hydrostatic (White & Wood, 1995)
Primitive equations (Richardson, 1922)
~~Acoustic~~
Lamb
Inertia-gravity
Rossby
Boussinesq
Anelastic (Ogura & Phillips)
Pseudo-incompressible (Durran ; Klein & Pauluis)
~~Acoustic~~
~~Lamb~~
Inertia-gravity
Rossby

Sound-proofing and degrees of freedom





- Euler equations : 5 prognostic fields (u,v,w, density, entropy)
  => linearized equations lead to a 5x5 determinant
  => degree-5 algebraic equation for ω(k)
  => 5 possible values (roots) for ω(k) :
  2 acoustic, 2 inertia-gravity, 1 Rossby

Hydrostatic :

- only 3 roots : inertia-gravity, Rossby
  => only 3 independent prognostic fields !
- Hydrostatic balance acts as a *constraint* that reduces the numbers of degrees of freedom
- **Which fields have become diagnostic ?**
- **How do I diagnose other fields ?**

Compressible hydrostatic dynamics :

are we ready to prognose ?

$$\begin{aligned}
\partial_z p(\rho, s) + \rho g &= 0 \\
\partial_t \rho + \partial_x(\rho u) + \partial_y(\rho v) + \partial_z(\rho w) &= 0 \\
\partial_t s + u\partial_x s + v\partial_y s + w\partial_z s &= 0 \\
\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u - fv + \frac{1}{\rho}\partial_x p &= 0 \\
\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + fu + \frac{1}{\rho}\partial_y p &= 0
\end{aligned}$$

- 4 prognostic equations for 3 degrees of freedom ??
- Hydrostatic balance yields density given entropy

- How do we diagnose vertical velocity w ?

- There must be an additional diagnostic relationship = *hidden constraint*
- In order to obtain hidden constraints, time-differentiate known constaints (here : hydrostatic balance)

Compressible hydrostatic dynamics : prognostic and diagnostic degrees of feedom

$$\partial_z \left[p\left(\rho, s\right) \right] + \rho g = 0$$

$$\begin{aligned} \partial_t s + w \partial_z s &= \dots \\ \partial_t \rho + \partial_z \left(\rho w \right) &= \dots \end{aligned}$$

$$\partial_z \rho \frac{\partial p}{\partial \rho} + \partial_z s \frac{\partial p}{\partial s} + g \partial_z \rho = 0$$

$$\partial_z \left(\partial_z \left(\rho w \right) \frac{\partial p}{\partial \rho} + w \partial_z s \frac{\partial p}{\partial s} \right) + g \partial_z \left(\rho w \right) = \dots$$

speed of sound

$$\frac{\partial p}{\partial \rho} = c^2$$

$$\partial_z \left(\rho c^2 \partial_z w \right) = \dots$$

Time-differentiating hydrostatic balance
=> diagnostic equation for vertical velocity

- First obtained by Richardson (1922) but using pressure as a prognostic variable
- Ooyama (1990) : prognosing entropy => « neater form » of Richardson's equation
- Dubos & Tort (2014) : General form for quasi-hydrostatic systems in curvilinear coordinates

Compressible hydrostatic dynamics :

are we ready to prognose ?

$$
\begin{aligned}
\partial_z p(\rho, s) + \rho g &= 0 \\
\partial_z \left(\rho c^2 \partial_z w \right) &= \ldots \\
\partial_t s + u\partial_x s + v\partial_y s + w\partial_z s &= 0 \\
\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u - fv + \frac{1}{\rho}\partial_x p &= 0 \\
\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + fu + \frac{1}{\rho}\partial_y p &= 0
\end{aligned}
$$

- Hydrostatic balance yields density given entropy
- Richardson's equation yields vertical velocity
- 3 prognostic equations for 3 degrees of freedom

- Typical compressible hydrostatic models do not work this way
- The reason can be found by having a closer look at *hydrostatic adjustment*

Hydrostatic adjustment

or : how nature imposes hydrostatic balance

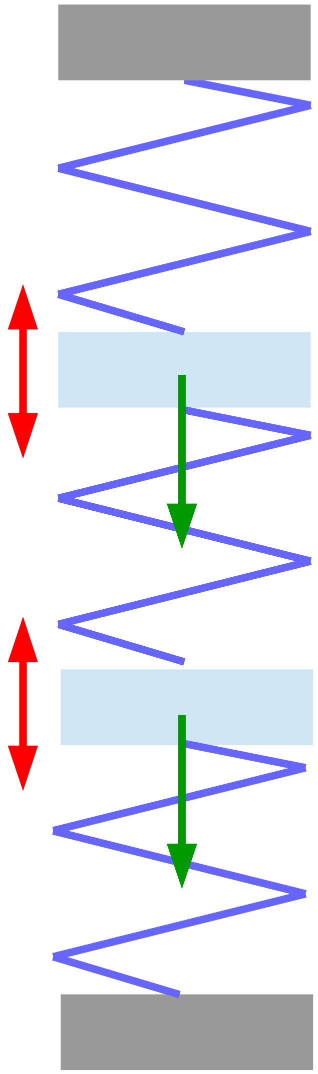
Consider

a horizontally homogeneous atmosphere
initial profile ρ(z), s(z) not hydrostatically balanced
what happens ?

Mechanical analogy :

a vertical stack of masses subject to gravity
and coupled by springs

- Vertical forces initially unbalanced
- Vertical acceleration, vertical displacements
- Oscillations / *acoustic* waves
- Until a balance is reached eventually
- Balanced state minimizes *mechanical energy*



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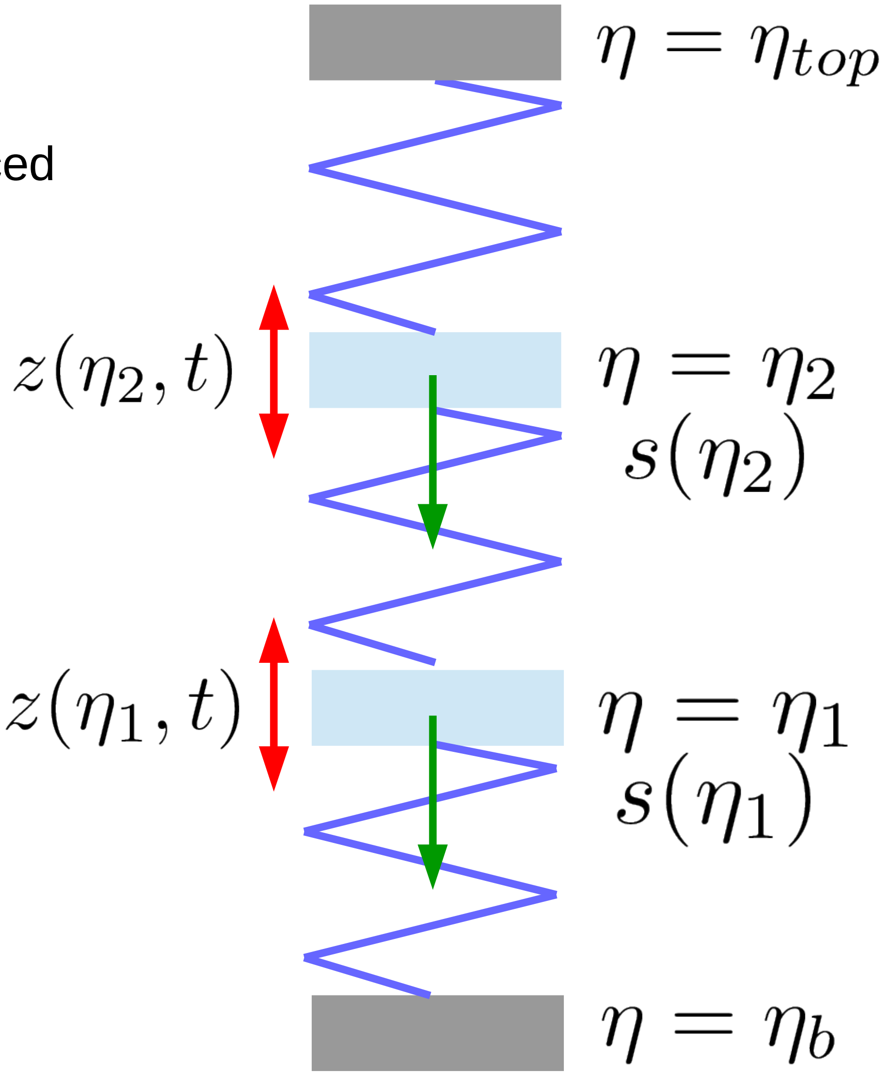

Mechanical analogy :

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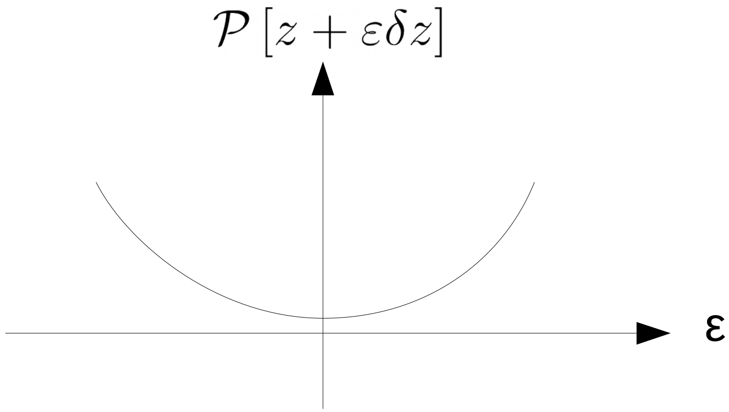
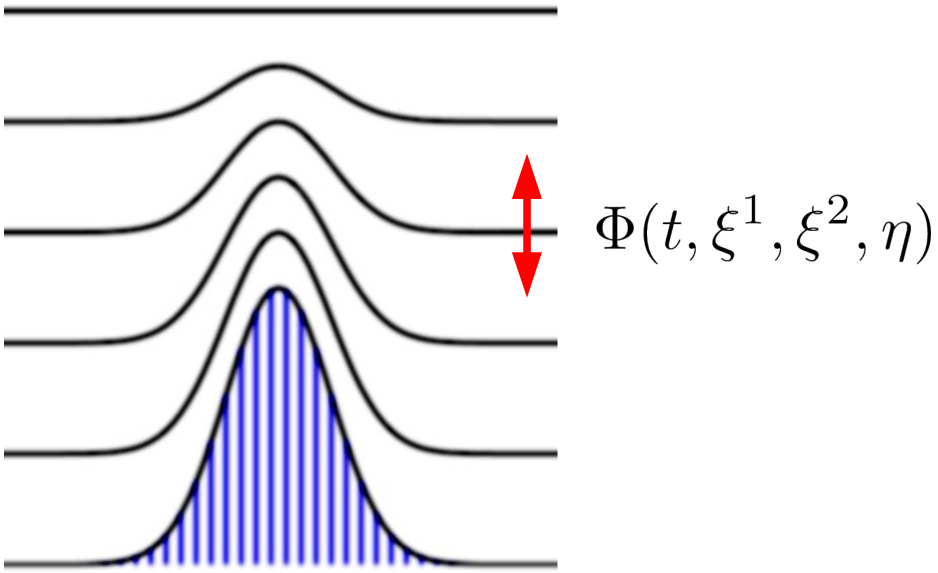
Hydrostatic adjustment and vertical coordinate





- Hydrostatic adjustment : 1D, well-posed, nonlinear elliptic problem where *the altitude z of air parcels is the unknown*
- In a hydrostatic model, a hydrostatic adjustment occurs at each time step

suggests to :

- let model layers « float » vertically
- let altitude z be a *field* rather than coordinate
- Non-Eulerian vertical coordinate

$$\partial_t\mu+\partial_1(\mu u^1)+\partial_2(\mu u^2)+\partial_\eta(\mu\dot\eta)=0$$
$$\partial_t s+u^1\partial_1 s+u^2\partial_2 s+\dot\eta\partial_\eta s=0$$

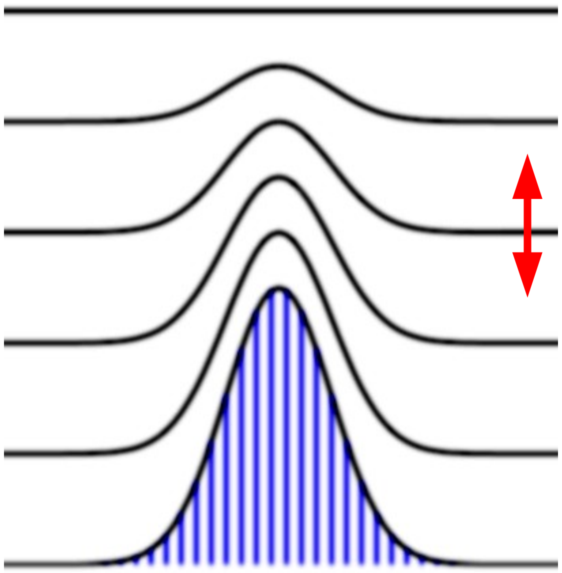
Procedure to *diagnose* $\dot\eta$ depends on specific definition of vertical coordinate. This definition is purely *kinematic*.

Generalized vertical coordinates

Isentropic vertical coordinate $s = s(\eta)$

- Entropy becomes *diagnostic*
- Vertical « velocity » given by heating

- Ground usually not an isentrope ...
- Ocean surface usually not an isopycnal...



$\Phi(t, \xi^1, \xi^2, \eta)$

Purely isentropic/isopycnal coordinate not used in practice

$$\partial_t \mu + \partial_1(\mu u^1) + \partial_2(\mu u^2) = 0$$
$$s = s(\eta)$$

Generalized vertical coordinates

$\boxed{\text{Lagrangian vertical coordinate } \dot{\eta} = 0}$

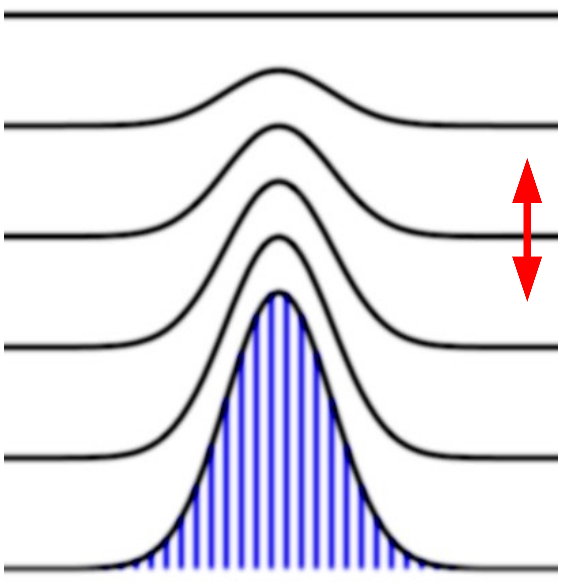
- simple
- initial altitude of layers can be chosen arbitrarily
- layers do not exchange mass, entropy

  => they follow the flow

  => layers tend to *fold* and *cross*

solution :

- exploit arbitraryness of altitude
- *vertical remap* before layers fold/cross



$$\Phi(t, \xi^1, \xi^2, \eta)$$

$$\partial_t \mu + \partial_1(\mu u^1) + \partial_2(\mu u^2) = 0$$

$$\partial_t s + u^1 \partial_1 s + u^2 \partial_2 s = 0$$

Generalized vertical coordinates

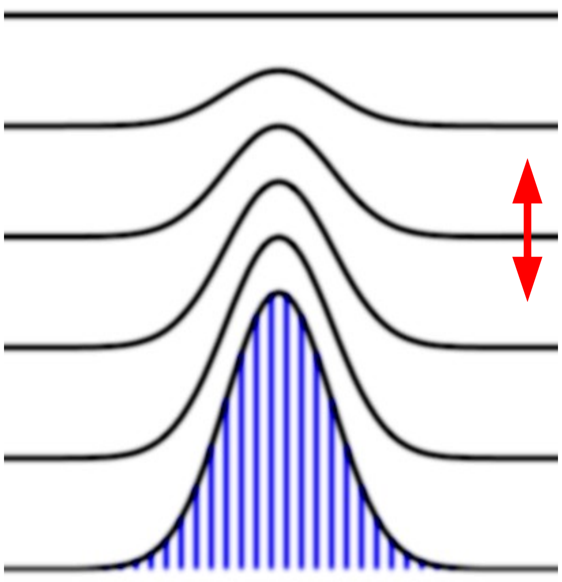
Hybrid mass-based coordinate

- each layer contains a prescribed fraction of the total mass of the column + fixed amount

$$M\left(t,\xi^1,\xi^2\right) \equiv \int \mu \mathrm{d}\eta$$

$$\mu = a(\eta)M + b(\eta)$$

- Layers exchange mass in order to respect this prescription
- There is always some mass in each layer, so layers never fold/cross



$\Phi(t,\xi^1,\xi^2,\eta)$

- *Diagnose mu from total column mass M*
- *Prognose M*
- *Diagnose dmu/dt*
- *Diagnose eta_dot*

$$\mu = a(\eta)M + b(\eta)$$

$$\partial_t M + \partial_1 \int \mu u^1 \mathrm{d}\eta + \partial_2 \int \mu u^2 \mathrm{d}\eta = 0$$

$$\partial_t \mu = A(\eta)\partial_t M$$

$$\partial_t \mu + \partial_1(\mu u^1) + \partial_2(\mu u^2) + \partial_\eta(\mu\dot{\eta}) = 0$$

Generalized vertical coordinates

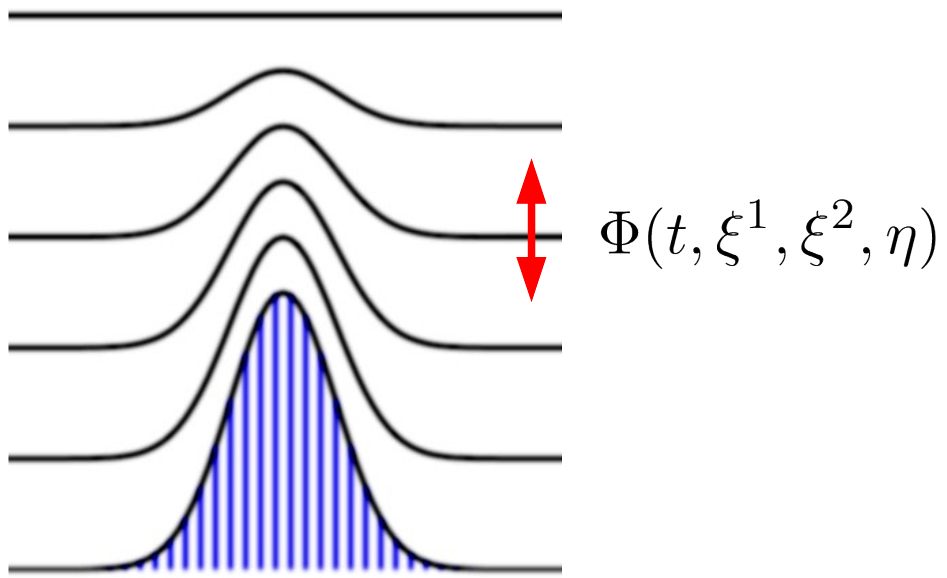
Hybrid pressure coordinate

Special form of mass-based, if we are solving **traditional hydrostatic** equations

$$\partial_\eta p = -\mu g$$

$$M\left(t, \xi^1, \xi^2\right) \equiv \int \mu \mathrm{d}\eta \;\Rightarrow p_s = p_{top} + gM$$

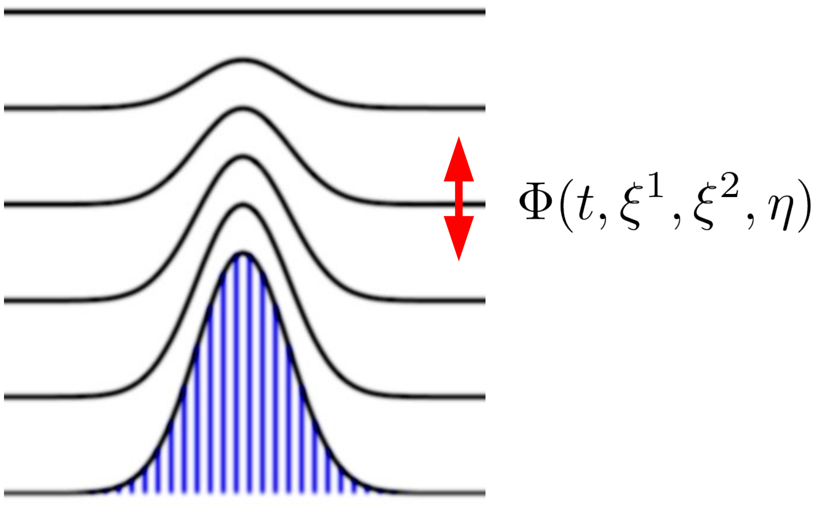
$$\mu = a(\eta)M + b(\eta) \quad \Rightarrow \boxed{p = A(\eta)p_s + B(\eta)}$$



- Prognose surface pressure rather than M
- Diagnose pressure straightforwardly
  No need to solve hydrostatic balance
- Hybrid coefficient A can be chosen close to 0 so that upper layers are nearly isobars

Recap : hydrostatic dynamics, generalized vertical coordinates & prognostic variables

- a hydrostatic adjustment occurs at each time step
- altitude z : time-dependent diagnostic field rather than coordinate
- Non-Eulerian vertical coordinate



Hybrid mass-based coordinate

kinematics

- *Diagnose pseudo-density mu from total column mass M*
- *Prognose M*
- *Diagnose dmu/dt*
- *Diagnose eta_dot*
- *Prognose entropy*

dynamics

- Hydrostatic adjustment => geopotential
- Prognose momentum

Lagrangian coordinate

kinematics

- *Prognose pseudo-density mu*
- *Prognose entropy*
- *If needed, vertical remap*

dynamics

- Hydrostatic adjustment => geopotential
- Prognose momentum

References

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Kasahara (1974)	Mon. Wea. Rev.**102**: 509-522
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