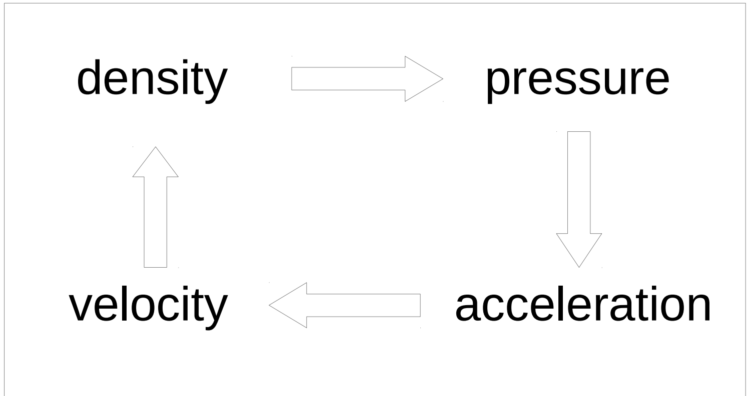
Boussinesq approximations

- (In)compressiblity, acoustic waves, and pressure
- Boussinesq approximation : principle and variants
- Accuracy of wave propagation

(In)compressibility, acoustic waves and pressure

The equation of state yields pressure given density and specific entropy (and moisture / salinity)

What happens if the fluid is *incompressible* ?



Propagation of acoustic waves

- Breaks the pressure-density feedback loop
- **Suppresses acoustic waves**
- But how do we determine pressure ??

$$h = h_0(s,r) + \alpha(s,r)(p - p_0)$$

$$L = \cdots + p\left(\frac{1}{\rho} - \alpha(s)\right)$$

Reminder : specific volume $\alpha \equiv \frac{1}{\rho}$

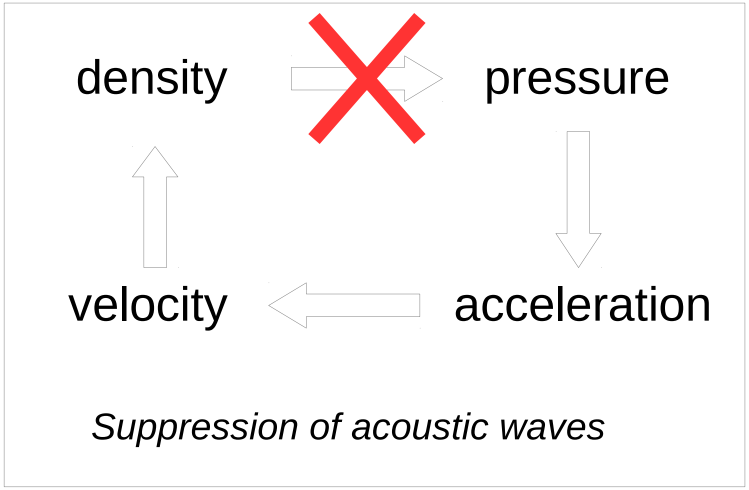
$$\frac{D}{Dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{\partial L}{\partial \mathbf{x}} - \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) = 0$$

$$\frac{\partial L}{\partial p} = 0$$

$$L(\mathbf{x}, \dot{\mathbf{x}}, \rho, s, p) = \ldots - h\,(p, s) + \frac{p}{\rho}$$

(In)compressibility, acoustic waves and pressure

- Incompressibilty **contrains** density to depend on entropy, salinity
- The Lagrangian is linear in pressure : **pressure is a Lagrange multiplier** enforcing the constraint ; it is must be determined by additional, **hidden constraints,** to be discovered



Known constraint (incompressibility) :

$$\rho\alpha(s,q) = 1$$

Using kinematics (transport) :

$$\frac{D\rho}{Dt} + \rho \operatorname{div}\dot{\mathbf{x}} = 0$$

$$\frac{\partial}{\partial t}$$

yields a

First hidden constraint :

$$\operatorname{div}\dot{\mathbf{x}} = 0$$

Using dynamics :

$$\frac{\partial \dot{\mathbf{x}}}{\partial t} + \frac{1}{\rho}\nabla p + \dots = 0$$

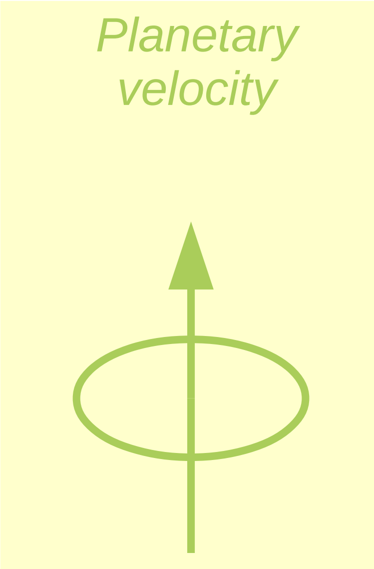
$$\frac{\partial}{\partial t}$$

yields a

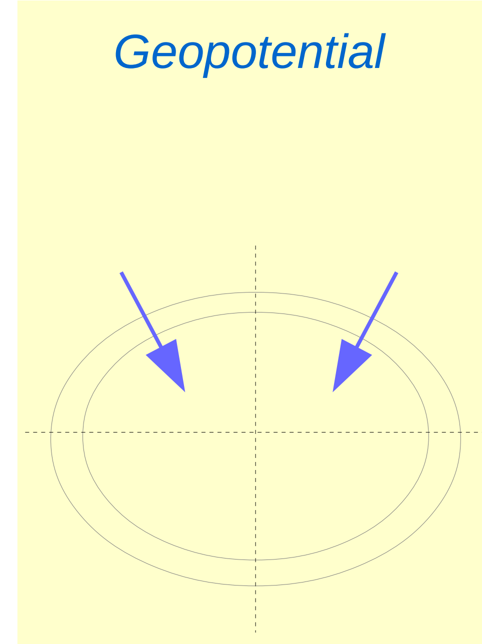
Second hidden constraint :

$$\nabla \cdot \frac{1}{\rho}\nabla p = \dots$$

This ***elliptic*** problem yields p but may be **hard/expensive** to solve.



$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\mathbf{R} \times \dot{\mathbf{x}} + \nabla\Phi + \frac{1}{\rho}\nabla p = 0$$



Basic idea of Boussinesq approximations :
pressure remains close to a fixed reference profile

$$p = \overline{p}(\Phi) + p'$$

$$\overline{\rho}(\Phi) \equiv -\frac{\partial \overline{p}}{\partial \Phi}$$

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p' = \underbrace{-\left(\frac{\overline{\rho}}{\rho} - 1\right)\mathbf{g}}_{\text{Buoyancy force}}$$

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$$p = \overline{p}(\Phi) + p'$$

$$\overline{\rho}(\Phi) \equiv -\frac{\partial \overline{p}}{\partial \Phi}$$

$$\overline{\rho} = \rho(\overline{p}, \overline{s})$$

$$\underbrace{\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p'}_{\text{Inertia}} = \underbrace{-\left(\frac{\overline{\rho}}{\rho} - 1\right)\mathbf{g}}_{\text{Buoyancy}}$$

Inertia

Buoyancy

$$\rho(\overline{p} + p', s) = \underbrace{\rho(\overline{p}, \overline{s})}_{\overline{\rho}(\Phi)} + \underbrace{\rho(\overline{p}, s)}_{\rho^*(\Phi, s)} - \rho(\overline{p}, \overline{s}) + \underbrace{\rho(\overline{p} + p', s) - \rho(\overline{p}, s)}_{\simeq \rho' = c^{-2} p'}$$

Reference density varies with altitude / depth

Warmer air rises, colder water sinks

Density fluctuates due to pressure variations caused by flow (dynamic pressure)

Basic idea of Boussinesq approximations :
pressure remains close to a fixed reference profile

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p' = -\left(\frac{\overline{\rho}}{\rho} - 1\right)\mathbf{g}$$

Inertia *Buoyancy*

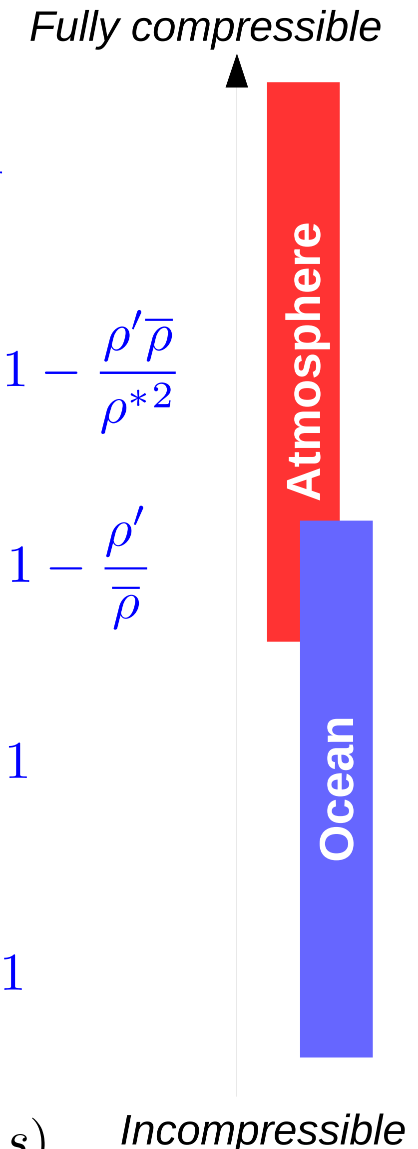
$$\rho(\overline{p} + p', s) = \quad \overline{\rho}(\Phi) \qquad\qquad \simeq \rho_0 = cst$$

$$+\rho(\overline{p}, s) - \rho(\overline{p}, \overline{s}) \qquad\qquad \simeq \rho(p_0, s) - \rho(p_0, \overline{s})$$

$$+\rho(\overline{p} + p', s) - \rho(\overline{p}, s) \qquad\qquad \simeq c^{-2}p' \quad \simeq 0$$

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p' = -b\mathbf{g}$$

- Exact : $\rho = \rho(p, s)$ $\quad b = \frac{\overline{\rho}}{\rho} - 1$
- Pseudo-incompressible : $\rho = \rho^*(\Phi, s)$ $\quad b = \frac{\overline{\rho}}{\rho^*} - 1 - \frac{\rho'\overline{\rho}}{\rho^{*2}}$
- Anelastic : $\rho = \overline{\rho}(\Phi)$ $\quad b = \frac{\overline{\rho}}{\rho^*} - 1 - \frac{\rho'}{\overline{\rho}}$
- Depth-dependent Boussinesq : $\rho = \rho_0 = cst$ $\quad b = \frac{\overline{\rho}}{\rho^*} - 1$
- Simple Boussinesq : $\rho = \rho_0 = cst$ $\quad b = \frac{\overline{\rho}}{\rho_0^*} - 1$

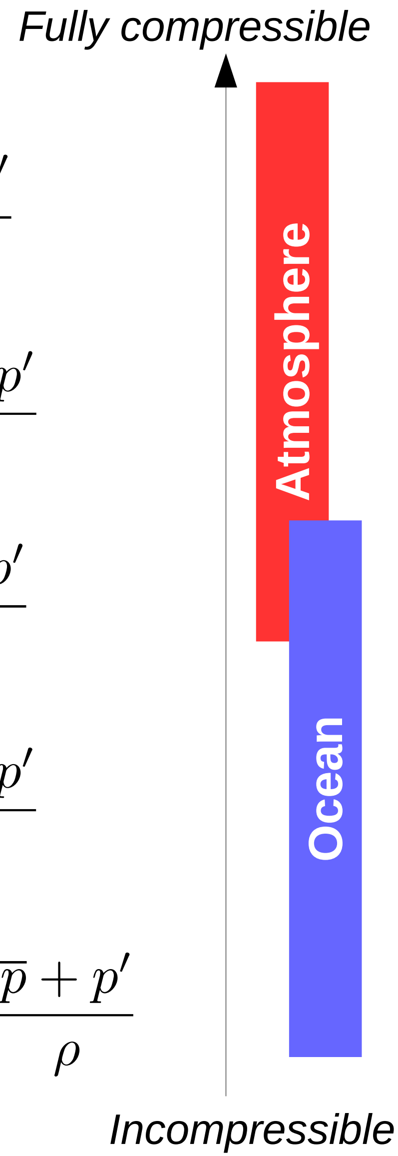


$$\rho_0^*(s) \equiv \rho(p_0, s)$$

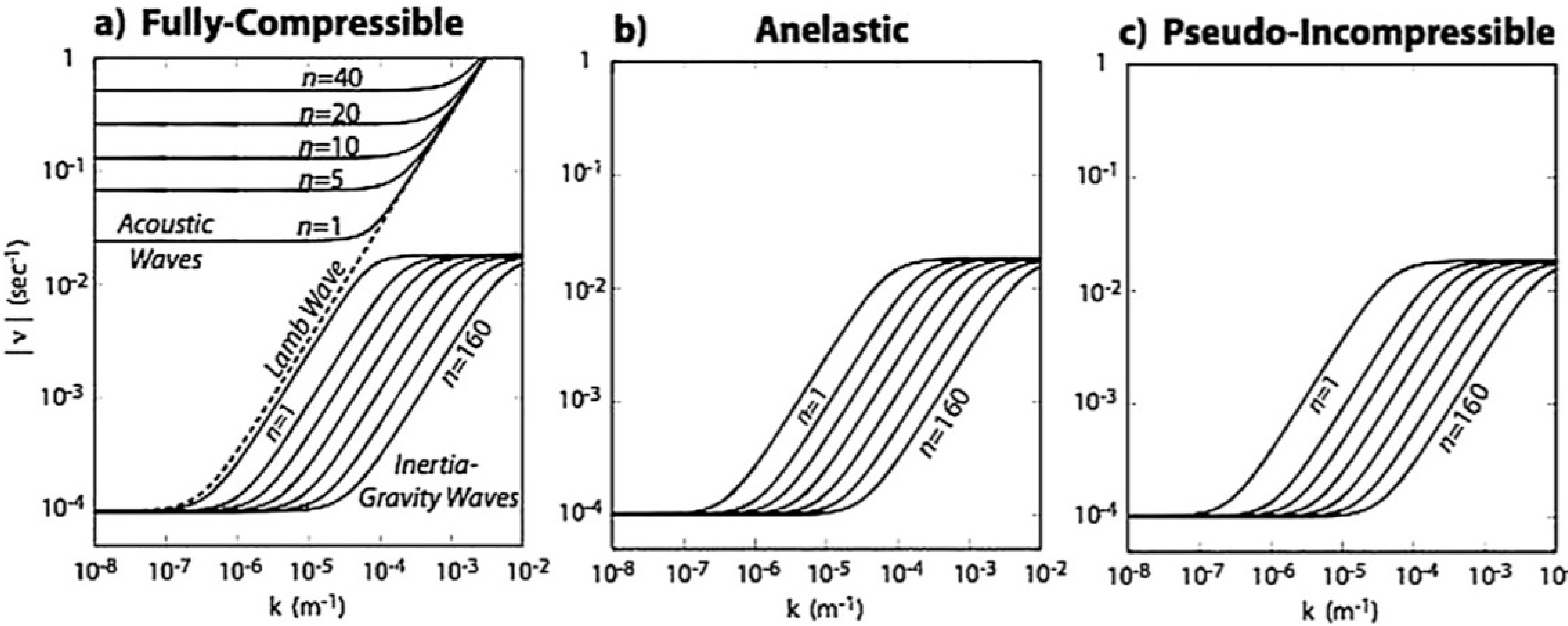
All the above combinations conserve energy/momentum/potential vorticity.

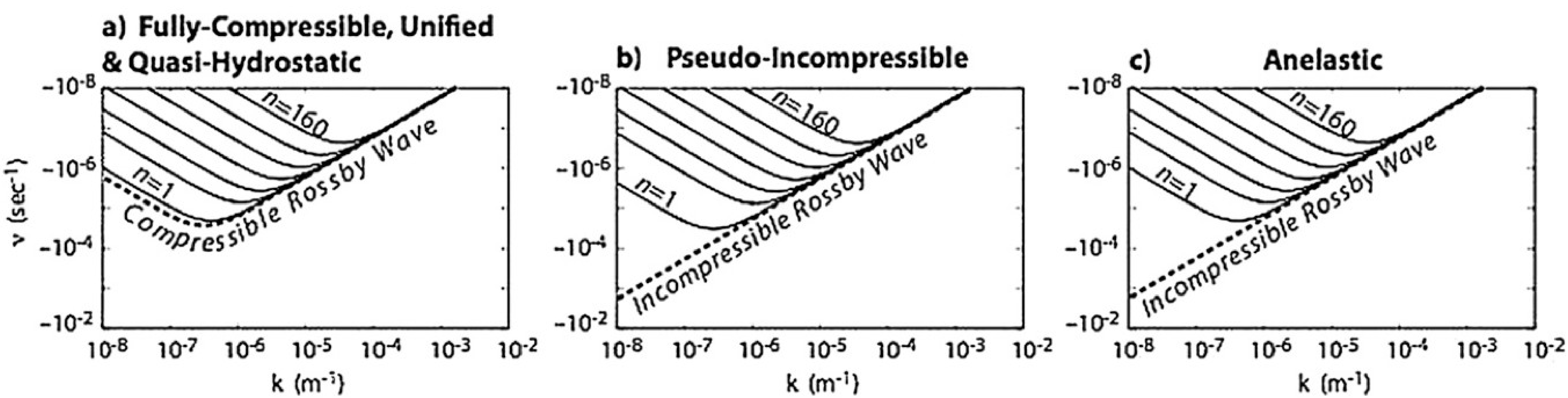
$$\frac{D}{Dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{\partial L}{\partial \mathbf{x}} - \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) = 0 \qquad \frac{\partial L}{\partial p'} = 0 \qquad L = K(\mathbf{x}, \dot{\mathbf{x}}) - E_p(\mathbf{x}, \rho, s, p')$$

- Exact : $$E_p = \Phi + h(\overline{p} + p', s) - \frac{\overline{p} + p'}{\rho}$$
- Pseudo-incompressible : $$E_p = \Phi + h(\overline{p}, s) + \frac{p'}{\rho^*} - \frac{\overline{p} + p'}{\rho}$$
- Anelastic : $$E_p = \Phi + h(\overline{p}, s) + \frac{p'}{\overline{\rho}} - \frac{\overline{p} + p'}{\rho}$$
- Depth-dependent Boussinesq : $$E_p = \Phi + h(\overline{p}, s) + \frac{p'}{\rho_0} - \frac{\overline{p} + p'}{\rho}$$
- Simple Boussinesq : $$E_p = \Phi\left(1 - \frac{\rho_0}{\rho_0^*(s)}\right) + \frac{p'}{\rho_0} - \frac{\overline{p} + p'}{\rho}$$



Static energy = Montgomery potential


a) Fully-Compressible
b) Anelastic
c) Pseudo-Incompressible
|ν| (sec⁻¹)
k (m⁻¹)
n=40
n=20
n=10
n=5
n=1
Acoustic Waves
Lamb Wave
n=1
n=160
Inertia-Gravity Waves
n=1
n=160
n=1
n=160


a) Fully-Compressible, Unified & Quasi-Hydrostatic
b) Pseudo-Incompressible
c) Anelastic
ν (sec⁻¹)
k (m⁻¹)
n=160
n=1
Compressible Rossby Wave
n=160
n=1
Incompressible Rossby Wave
n=160
n=1
Incompressible Rossby Wave

Pros / cons

All

- Acoustic waves are filtered and do not limit the time step
- A 3D elliptic problem must be solved (hard/expensive)
- Can be combined with the hydrostatic approximation => 1D elliptic problem (easy)

Pseudo-incompressible

- Very accurate, except for long barotropic Rossby waves (too fast)
  => not for global atmospheric models (definition of a reference profile also an issue)
- Elliptic problem has time-dependent coefficients

Anelastic

- Same issues with long barotropic Rossby waves
- Formally, accurate if density close to an adiabatic profile => OK for convection
- In practice, still accurate quite far from neutral stability
  => good for regional nonhydrostatic modelling

Depth-dependent Boussinesq

- Accurate for nearly-incompressible fluid (water)
- Used in most realistic ocean models

Simple Boussinesq

- Not accurate enough for realistic ocean modelling
  But good enough for process studies and idealized modelling
- Conceptual model for atmospheric flow
- Amenable to analytic solutions (with linearized equation of state)

References

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