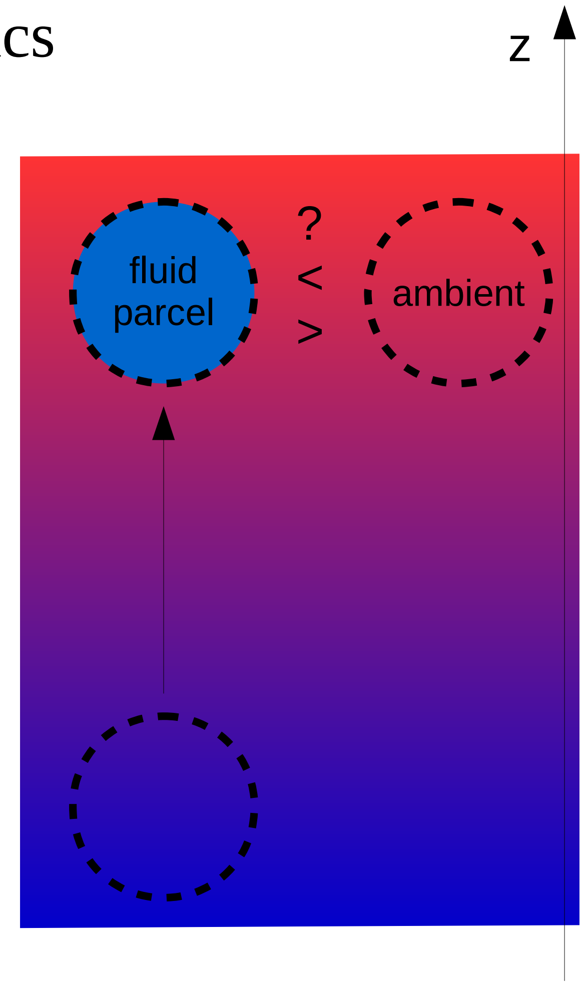
Dynamical approximations

Thomas Dubos
E. Polytechnique / LMD / IPSL

Energy
Momentum
Potential vorticity

Inertial Cartesian Eulerian	Rotating Cartesian Eulerian	Rotating Curvilinear Eulerian	Rotating Curvilinear Eulerian	Rotating Curvilinear non-Eulerian
Thermodynamics *Gravity*	*Planetary velocity* *Geopotential* *Coriolis*	*Kinematics* *Covariant* *Contravariant*	*Spherical geoid* *Shallow-fluid* *Traditional* *Boussinesq/anelastic* *Hydrostatic*	*Terrain-following* *Lagrangian* *Isentropic/isopycnal* *mass-/pressure-based*

Dynamical effects of thermodynamics



The atmosphere and ocean are *stratified* flows

Consider the simple question of the stability of a resting atmosphere/ocean

- Naive criterion : density decreases with altitude
- Time scale of buoyancy oscillations : Brunt-Vaisala frequency N

$$N^2 = -\frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z} > 0$$

- In fact : fluid parcels adjust rapidly and adiabatically their density to ambient pressure

$$\Rightarrow \qquad N^2 = -\frac{g}{\rho}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z} - \frac{\mathrm{d}p}{\mathrm{d}z}\frac{\partial\rho}{\partial p}\right)$$

density

$$\rho = \rho(s, p, q)$$

Mass fraction (salinity)

Specific entropy

temperature

pressure

$$\theta = T(s, p_0)$$

- atmospheric scientists use potential temperature

$$N^2 = -\frac{g}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}z} > 0$$

Important to have a *simple*, *correct* and *general* approach to thermodynamics

Thermodynamics of a compressible fluid

Systematic approach (Ooyama, 1990 ; Bannon, 2003 ; Feistel, 2008)

- state variables :
  - specific volume/pressure
  - specific entropy / temperature
  - mixing ratio / chemical potential (mixtures)
- All relations follow from the expression of a single thermodynamic potential

$$\mathrm{d}e = -p\mathrm{d}\alpha + T\mathrm{d}s + \mu\mathrm{d}q$$
$$\Rightarrow \quad e(\alpha, s, q)$$
$$\alpha \leftrightarrow p \qquad s \leftrightarrow T \qquad q \leftrightarrow \mu$$

Specific enthalpy $h(p, s, q) \equiv e + \alpha p$

Specific Gibbs function $g(p, T, q) \equiv h - Ts$

Pro :
- always energetically consistent
- general : variable cp, mixtures

Con :
- none
- unless you *really* hate thermodynamics

$$\mathrm{d}h = \alpha\mathrm{d}p + T\mathrm{d}s + \mu\mathrm{d}q$$
$$\mathrm{d}g = \alpha\mathrm{d}p - s\mathrm{d}T + \mu\mathrm{d}q$$

Ideal perfect gas

$$h(p,s) = h_0 - C_p T_0 + C_p T_0 \left(\frac{p}{p_0}\right)^{R/C_p} \exp\frac{s-s_0}{C_p}$$

$$T(p,s) = \frac{\partial h}{\partial s} = T_0 \left(\frac{p}{p_0}\right)^{R/C_p} \exp\frac{s-s_0}{C_p}$$

$$\mathrm{d}h = C_p \mathrm{d}T$$

$$\theta = T(p_0, s) = T_0 \exp\frac{s-s_0}{C_p}$$

$$\theta = T\left(\frac{p}{p_0}\right)^{R/C_p}$$

$$\alpha(p,s) = \frac{\partial h}{\partial p} = \frac{R}{C_p}\frac{h-(h_0 - C_p T_0)}{p}$$

$$p = \rho R T$$

Incompressible fluid

$$h(p, s, q) = h_0(s, q) + \alpha(s, q)(p - p_0)$$

$$\alpha = \frac{\partial h}{\partial p} = \alpha(s, q)$$

Density varies only due to heating / salinity

$$e = h - \alpha p = h_0 - \alpha(s, q) p_0$$

Internal energy is degenerate
=> use enthalpy to describe nearly incompressible fluids

$$T = \frac{\partial h}{\partial s} = \frac{\partial h_0}{\partial s} + \frac{\partial \alpha_0}{\partial s}(p - p_0)$$

$$\frac{\partial T}{\partial p} = \frac{\partial \alpha}{\partial s}$$

$$T(p_0, s) = \frac{\partial h_0}{\partial s}$$

Potential temperature

$$h(p_0, s, q) = h_0(s, q)$$

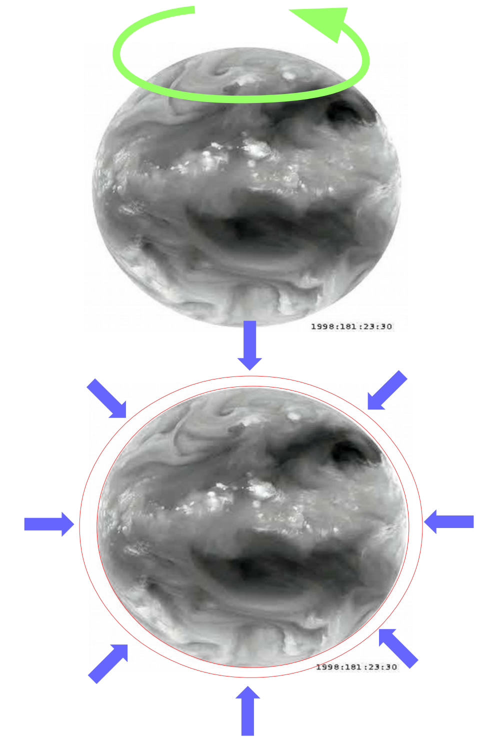
Potential enthalpy
~ conservative temperature

Lagrangian least action principle for fluid flow

(Eckart, 1960 ; Morrison, 1998)

$$\rho = \rho(p, s)$$

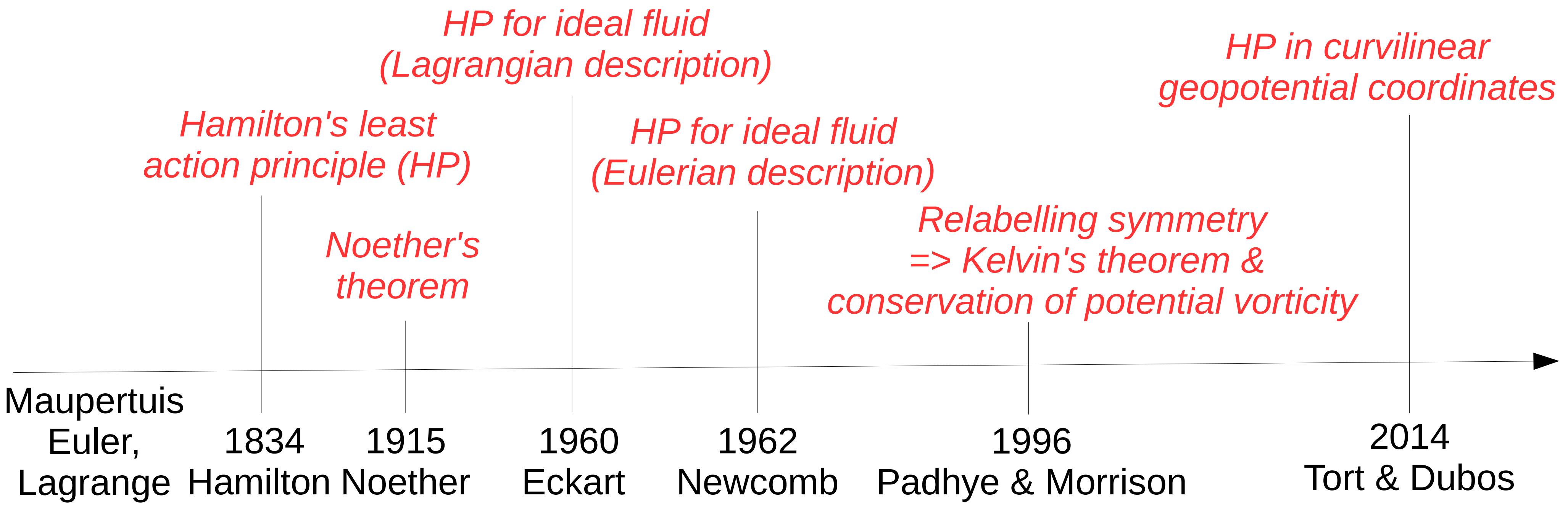
$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\mathbf{R} \times \dot{\mathbf{x}} + \nabla\Phi + \frac{1}{\rho}\nabla p = 0$$



$$\frac{D}{Dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{\partial L}{\partial \mathbf{x}} - \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) = 0$$

$$\frac{\partial L}{\partial p} = 0$$

$$L(\mathbf{x}, \dot{\mathbf{x}}, \rho, s, p) = \frac{\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}}{2} + \mathbf{R} \cdot \dot{\mathbf{x}} - \Phi - h\,(p, s) + \frac{p}{\rho}$$


HP for ideal fluid
(Lagrangian description)
HP in curvilinear
geopotential coordinates
Hamilton's least
action principle (HP)
HP for ideal fluid
(Eulerian description)
Relabelling symmetry
=> Kelvin's theorem &
conservation of potential vorticity
Noether's
theorem
Maupertuis
Euler,
Lagrange
1834
Hamilton
1915
Noether
1960
Eckart
1962
Newcomb
1996
Padhye & Morrison
2014
Tort & Dubos

$$\frac{D}{Dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{\partial L}{\partial \mathbf{x}} - \frac{1}{\rho}\nabla\left(\rho^2\frac{\partial L}{\partial\rho}\right) = 0$$

$$\mathbf{v} = \frac{\partial L}{\partial \dot{\mathbf{x}}}$$

conjugate momentum
= absolute velocity

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) - \nabla\left(\rho^2\frac{\partial L}{\partial\rho}\right) + \mathrm{div}\ [\rho\dot{\mathbf{x}}\otimes\mathbf{v}] = \rho\frac{\partial L}{\partial\mathbf{x}}$$

Flux-form momentum budget

$$E = \dot{\mathbf{x}}\cdot\mathbf{v} - L$$

energy

$$\frac{\partial}{\partial t}(\rho E) + \mathrm{div}\ \left[\left(E - \rho\frac{\partial L}{\partial\rho}\right)\rho\dot{\mathbf{x}}\right] = 0$$

Flux-form energy budget

$$B = \mathbf{v}\cdot\dot{\mathbf{x}} - \rho\frac{\partial L}{\partial\rho} - L$$

Bernoulli function

$$\frac{\partial\mathbf{v}}{\partial t} + \frac{\nabla\times\mathbf{v}}{\rho}\times\rho\mathbf{u} + \nabla B - T\nabla s = 0$$

Crocco's theorem = curl-form

$$q = \frac{1}{\rho}(\nabla\times\mathbf{v})\cdot\nabla s$$

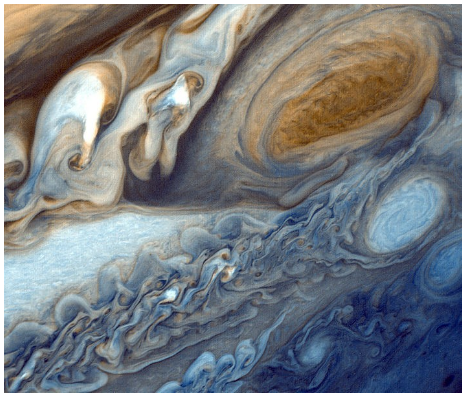
$$\frac{Dq}{Dt} = 0$$

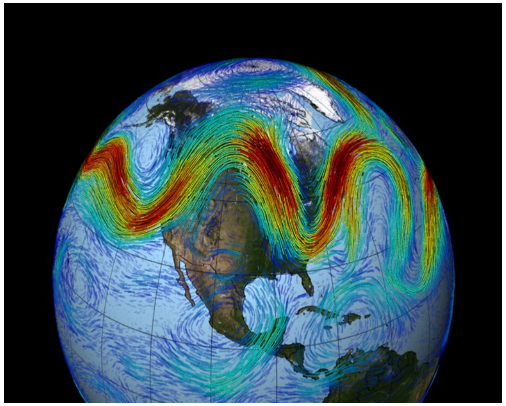
Potential vorticity

Characteristic scales

- *Velocity* : Sound $c \sim 340 m/s$ Wind $U \sim 30 m/s$
- *Time* : Buoyancy oscillations $N \sim g/c \sim 10^{-2} s^{-1}$ Coriolis $f \sim 10^{-4} s^{-1}$
- *Length* : **Scale height $H=c^2/g=10km$** **Rossby radius : $R=c/f \sim 1000$ km**

Mach number : $M=U/c \ll 1$ ***Scale separation : $f/N \sim H/R \ll 1$***



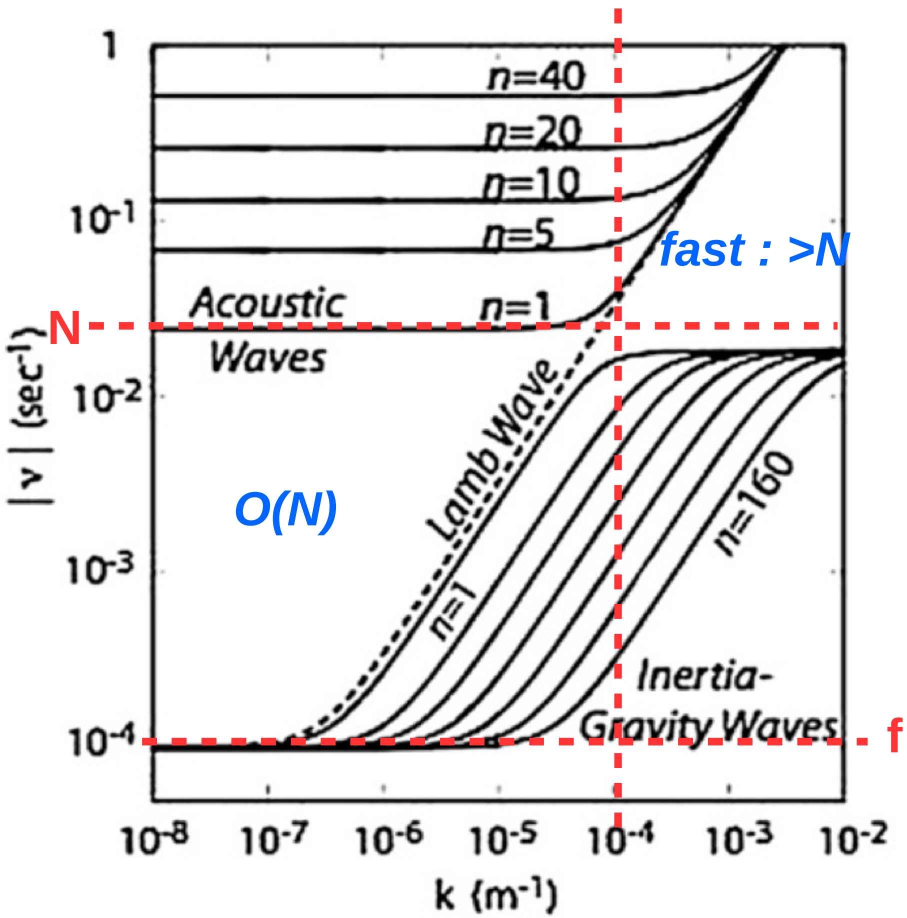


small-scale scale height mesoscale synoptic planetary

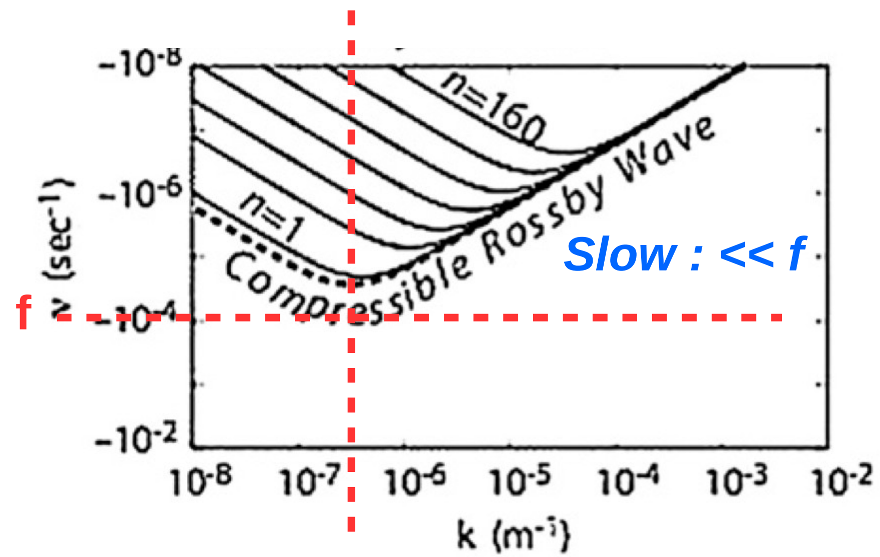
1 km 10 km 100 km 1000 km 10000 km

Classes of motion in a gravity-dominated, rotating, compressible flow :

Waves in an isothermal atmosphere at rest



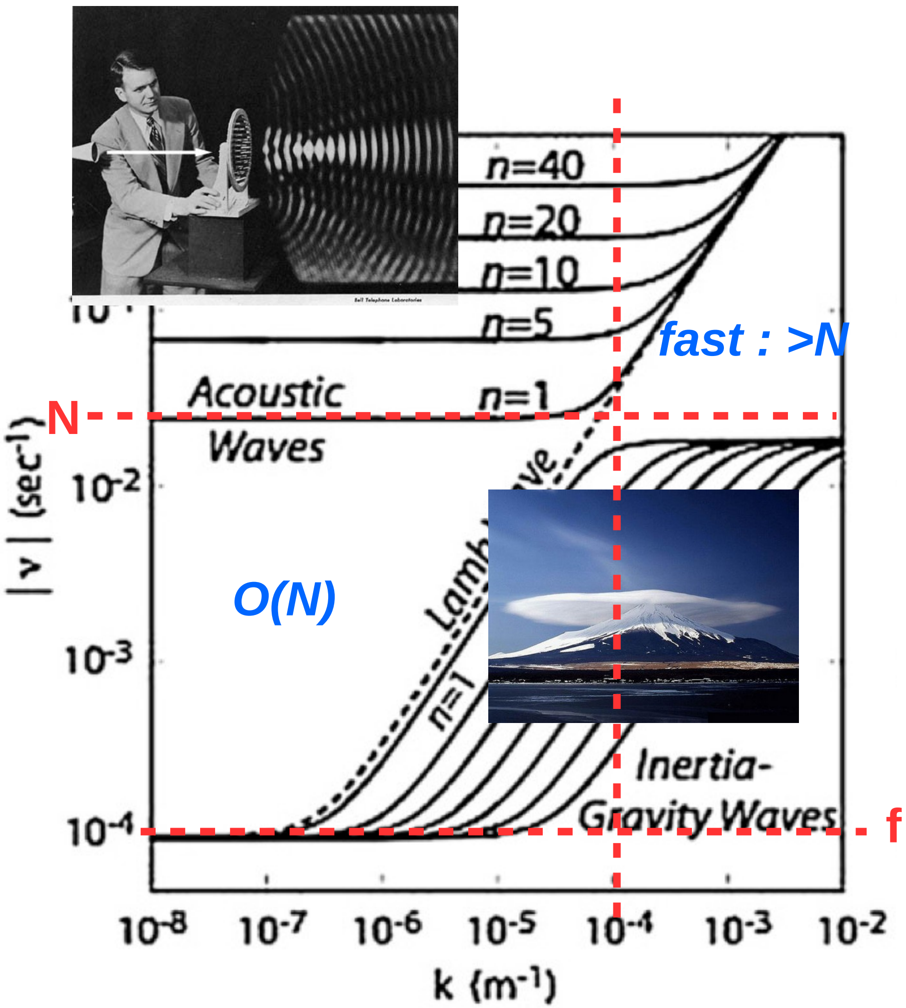
f-plane



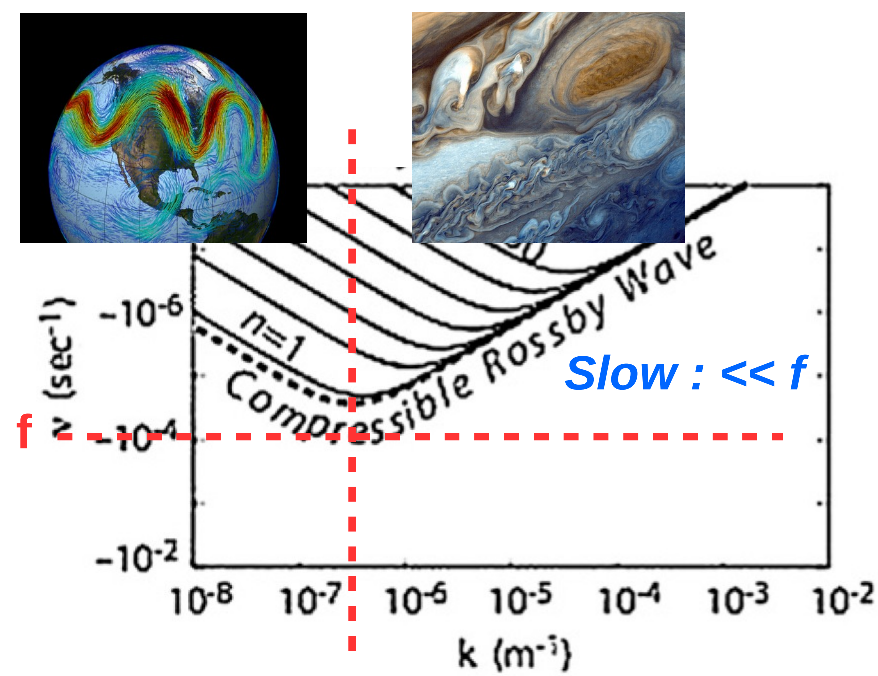
beta-plane

Classes of motion in a gravity-dominated, rotating, compressible flow :

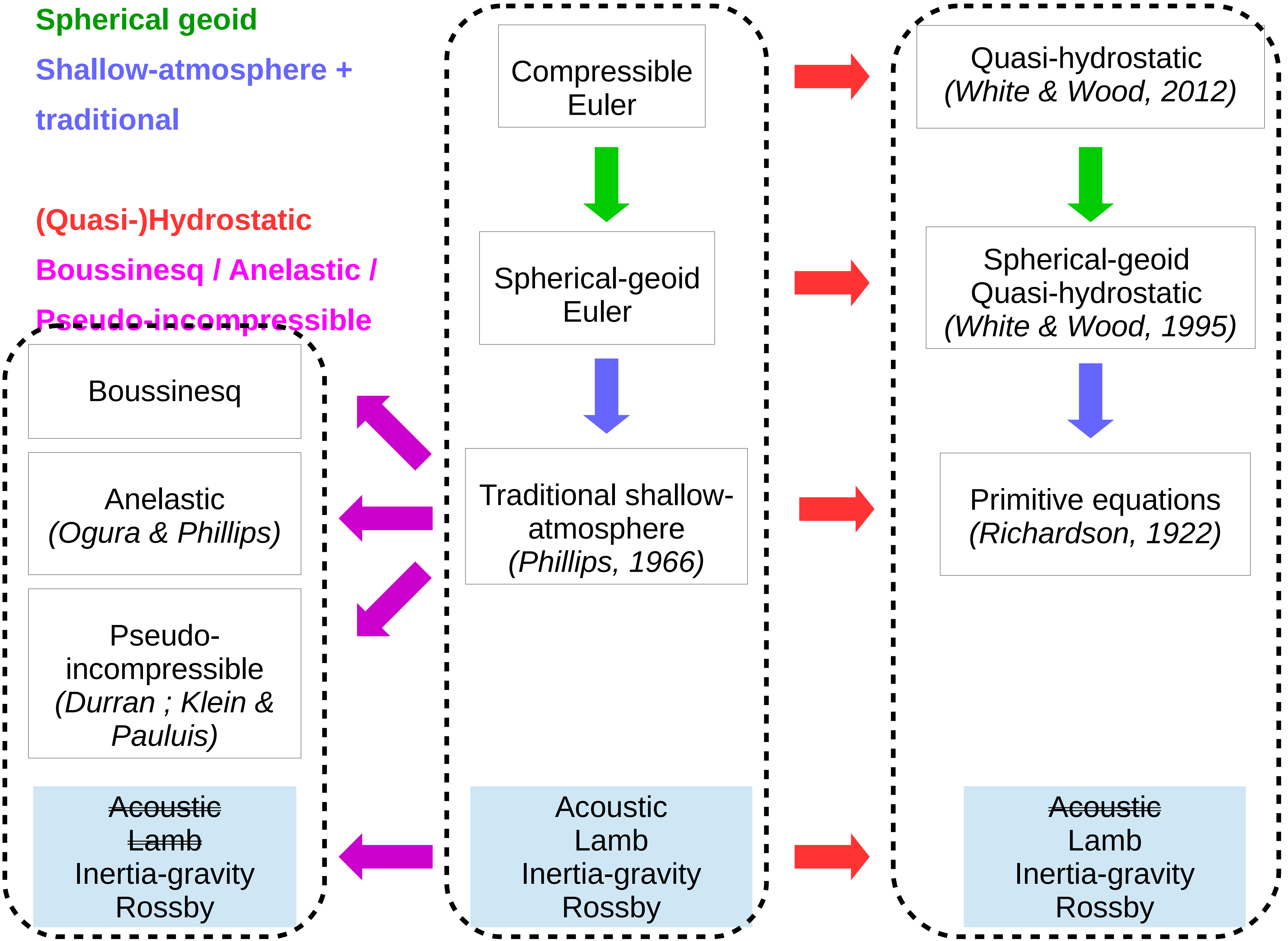
Waves in an isothermal atmosphere at rest



f-plane



beta-plane


Spherical geoid
Shallow-atmosphere + traditional
(Quasi-)Hydrostatic
Boussinesq / Anelastic / Pseudo-incompressible
Compressible Euler
Spherical-geoid Euler
Traditional shallow-atmosphere (Phillips, 1966)
Acoustic
Lamb
Inertia-gravity
Rossby
Quasi-hydrostatic (White & Wood, 2012)
Spherical-geoid Quasi-hydrostatic (White & Wood, 1995)
Primitive equations (Richardson, 1922)
~~Acoustic~~
Lamb
Inertia-gravity
Rossby
Boussinesq
Anelastic (Ogura & Phillips)
Pseudo-incompressible (Durran ; Klein & Pauluis)
~~Acoustic~~
~~Lamb~~
Inertia-gravity
Rossby

	NEMO	**ROMS**	**IFS/ARPEGE**	**MesoNH**	**WRF**	**EndGAME**	**LMDZ**	**DYNAMICO**
Geometry	SG+TSA	SG+TSA	SG+TSA	SG+TSA	SG+TSA	SG	SG+TSA	SG+TSA
Dynamics	HB	HB	FCE	A	FCE	FCE	HPE	HPE/(FCE)
Grid	CC	CC	LL	CC	CC	LL	LL	HEX
Disc. Dyn	FD	FV	SP	FD	FV	FD	FD	FD
Transport	FV	FV	SL	FV	FV	FV	FV	FV
Conserv.	M, E/Z	M		M	M	M	M, E/Z	M, E
Time	Split-EX	Split-EX	SI	EX	Split-HEVI	SI	EX	EX/HEVI
Helmholtz			Direct	Direct		Iter		

SG	Spherical-Geoid
TSA	Traditional Shallow-Atmosphere
FCE	Fully Compressible Euler
HPE	Hydrostatic Primitive Eq.
HB	Hydrostatic Boussinesq
A	Anelastic
EX	Explicit
SI	Semi-Implicit
Split	Split
HEVI	Horizontally Explicit, Vertically Implicit
Direct	Direct (spectral)
Iter	Iterative

CC	Cartesian Curvilnear
LL	Latitude-Longitude
HEX	Icosahedral-Hexagonal
FD	Finite Difference
FV	Finite Volume
FE	*Finite Element*
SE	*Spectral Element*
SL	Semi-Lagrangian
SP	Spectral
M	Mass and scalars
E	Energy
Z	Enstrophy