Modélisation Numérique de l'Océan et de l'Atmosphère

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Fluid motion and transport in a near-spherical shell

Contents

Transport in Cartesian coordinates

Conservation of mass and fluid velocity

Rien ne se perd, rien ne se crée, tout se transporte

Consider first

- a **single-component fluid**, e.g. pure water.
- with number of molecules per unit volume $N(t, \boldsymbol{x})$

Then the **conservation of number of molecules** in a domain $\mathcal{D}$ has integral and local formulations :

$$\boxed{\underbrace{\frac{\partial}{\partial t}\int_{\mathcal{D}} N \,\mathrm{d}^3\boldsymbol{x}}_{\text{molecules in domain } \mathcal{D}} + \underbrace{\int_{\partial\mathcal{D}} \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S}_{\text{molecules crossing the boundary}} = 0 \quad \Longleftrightarrow \quad \frac{\partial N}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F} = 0}$$

Given its density N and flux $\boldsymbol{F}$ we can define the **velocity** $\dot{\boldsymbol{x}}$ of the fluid:

$$\dot{\boldsymbol{x}} \equiv \frac{\boldsymbol{F}}{N} \quad \Longrightarrow \quad \frac{\partial N}{\partial t} + \boldsymbol{\nabla}\cdot(N\dot{\boldsymbol{x}}) = 0$$

Reminder: $\int_{\mathcal{D}} \boldsymbol{\nabla}\cdot\boldsymbol{F}\,\mathrm{d}^3\boldsymbol{x} = \int_{\partial\mathcal{D}} \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S$

Conservation of mass and fluid velocity

Consider now a **mixture** of molecules N_2, O_2, H_2O ...

- each with number per unit volume $N_n(t, \boldsymbol{x})$ and flux $\boldsymbol{F}_n(t, \boldsymbol{x})$
- and a rate per unit volume Σ_n of creation/destruction by chemistry, phase change

$$\implies \frac{\partial N_n}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F}_n = \underbrace{\Sigma_n}_{\text{production / destruction}} \qquad \text{where} \quad \overbrace{\sum_n \underbrace{M_n}_{\text{molar masses}} \Sigma_n = 0}^{\text{conservation of mass}}$$

Now define density as $\rho \equiv \sum_n M_n N_n$ Then we can *define* the fluid velocity $\dot{\boldsymbol{x}}$ as the barycentric velocity:

$$\boxed{\dot{\boldsymbol{x}} \equiv \frac{\sum_n M_n \boldsymbol{F}_n}{\rho} \qquad \implies \qquad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \dot{\boldsymbol{x}}) = 0}$$

For atmospheric applications, the air velocity is sometimes defined as the **velocity of dry air only**. In that case the budget for total mass (dry+moist) **has a r.h.s** which is a flux divergence.

Trajectories and Lagrangian derivatives

Now that we have a definition for the fluid velocity, we can

- define *fluid parcels* that "move with the fluid": their position $\boldsymbol{x}(t)$ is such that is velocity coincides with the fluid velocity

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \dot{\boldsymbol{x}}(t, \boldsymbol{x}(t))$$

- ask how a quantity $q(t, \boldsymbol{x})$ evolves "along a fluid trajectory":

$$\frac{\mathrm{d}}{\mathrm{d}t} q(t, \boldsymbol{x}(t)) = \frac{\partial q}{\partial t} + \dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla} q$$

Lagrangian / material derivative

$$\dot{q} = \frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla} q = \frac{\partial q}{\partial t} + \dot{x}\frac{\partial q}{\partial x} + \dot{y}\frac{\partial q}{\partial y} + \dot{z}\frac{\partial q}{\partial z}$$

Advection and diffusion

Now for each species n we can

- define its mass density, mass source and mass ratio:

$$\rho_n \equiv M_n N_n \qquad \sigma_n = M_n \Sigma_n \qquad q_n \equiv \frac{\rho_n}{\rho}$$

- split its mass flux $M_n \boldsymbol{F}_n$ into a part due to the fluid velocity and a a **diffusive flux**:

$$\underbrace{M_n \boldsymbol{F}_n}_{\text{total}} = \underbrace{M_n N_n \dot{\boldsymbol{x}}}_{\text{advective}} + \underbrace{\boldsymbol{f}_n}_{\text{diffusive}}$$

so that:

$$\boxed{\begin{aligned} \overbrace{\frac{\partial \rho_n}{\partial t} + \boldsymbol{\nabla} \cdot \rho_n \dot{\boldsymbol{x}}}^{\text{reversible / resolved}} &= \overbrace{\sigma_n - \boldsymbol{\nabla} \cdot \boldsymbol{f}_n}^{\text{irreversible / unresolved}} \\ \rho \frac{\mathrm{D} q_n}{\mathrm{D} t} &= \sigma_n - \boldsymbol{\nabla} \cdot \boldsymbol{f}_n \end{aligned}}$$

Consequences of the transport equations

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \quad \Longrightarrow \quad \frac{\mathrm{D}}{\mathrm{D}t} F(q) = 0$$

$$q = cst \text{ at } t = 0 \quad \Longrightarrow \quad q = cst \text{ at } t > 0$$

$$q \geq 0 \text{ at } t = 0 \quad \Longrightarrow \quad q \geq 0 \text{ at } t > 0$$

$$\frac{\mathrm{D}q_1}{\mathrm{D}t} = \ldots = \frac{\mathrm{D}q_N}{\mathrm{D}t} = 0 \quad \Longrightarrow \quad \frac{\mathrm{D}}{\mathrm{D}t} F(q_1, \ldots, q_N) = 0$$

$$F(q_1, \ldots, q_N) = 0 \text{ at } t = 0 \quad \Longrightarrow \quad F(q_1, \ldots, q_N) = 0 \text{ at } t > 0$$

Recap: Transport in Cartesian coordinates

Lagrangian / material derivative

$$\dot{q} = \frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla} q$$

Conservative form = Flux form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \dot{\boldsymbol{x}} = 0$$

$$\frac{\partial \rho_n}{\partial t} + \boldsymbol{\nabla} \cdot \rho_n \dot{\boldsymbol{x}} = \sigma_n - \boldsymbol{\nabla} \cdot \boldsymbol{f}_n$$

Advective form

$$\rho \frac{\mathrm{D} q_n}{\mathrm{D} t} = \sigma_n - \boldsymbol{\nabla} \cdot \boldsymbol{f}_n$$

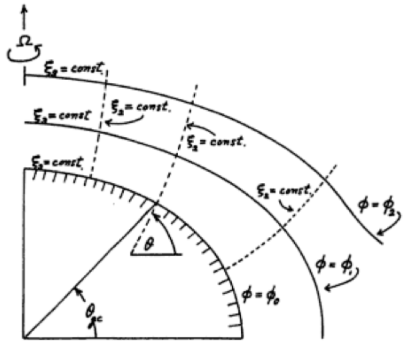
Implications

$$\frac{\mathrm{D} q}{\mathrm{D} t} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d} t} \int_{\mathcal{D}} \rho q \, \mathrm{d}^3 \boldsymbol{x} = 0$$

$$\frac{\mathrm{D} q_n}{\mathrm{D} t} = 0 \quad \Longrightarrow \quad \frac{\mathrm{D}}{\mathrm{D} t} F(q_1, \ldots, q_N) = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d} t} \int_{\mathcal{D}} \rho F(q_1, \ldots, q_N) \, \mathrm{d}^3 \boldsymbol{x} = 0$$

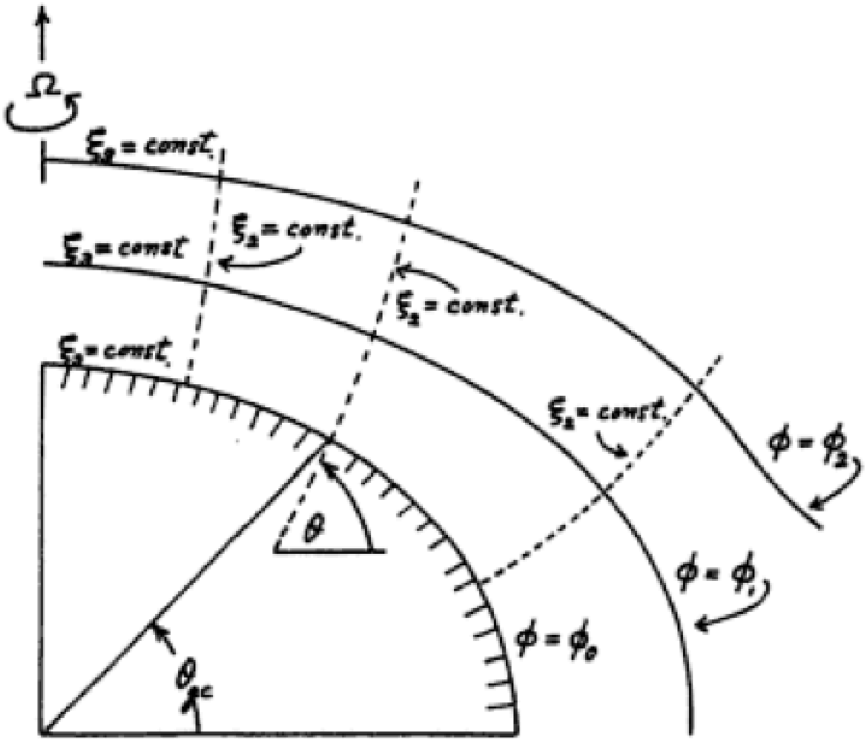
(Boundary terms omitted)

Transport in curvilinear coordinates



Surfaces of constant ξ_2 and constant ξ_3

Lagrangian derivative in curvilinear coordinates



Surfaces of constant ξ_2 and constant ξ_3

The geometry of the Earth is better suited to *curvilinear* coordinates:

- for instance spherical coordinates λ, φ, r
- in general ξ^1, ξ^2, ξ^3
- by the chain rule:

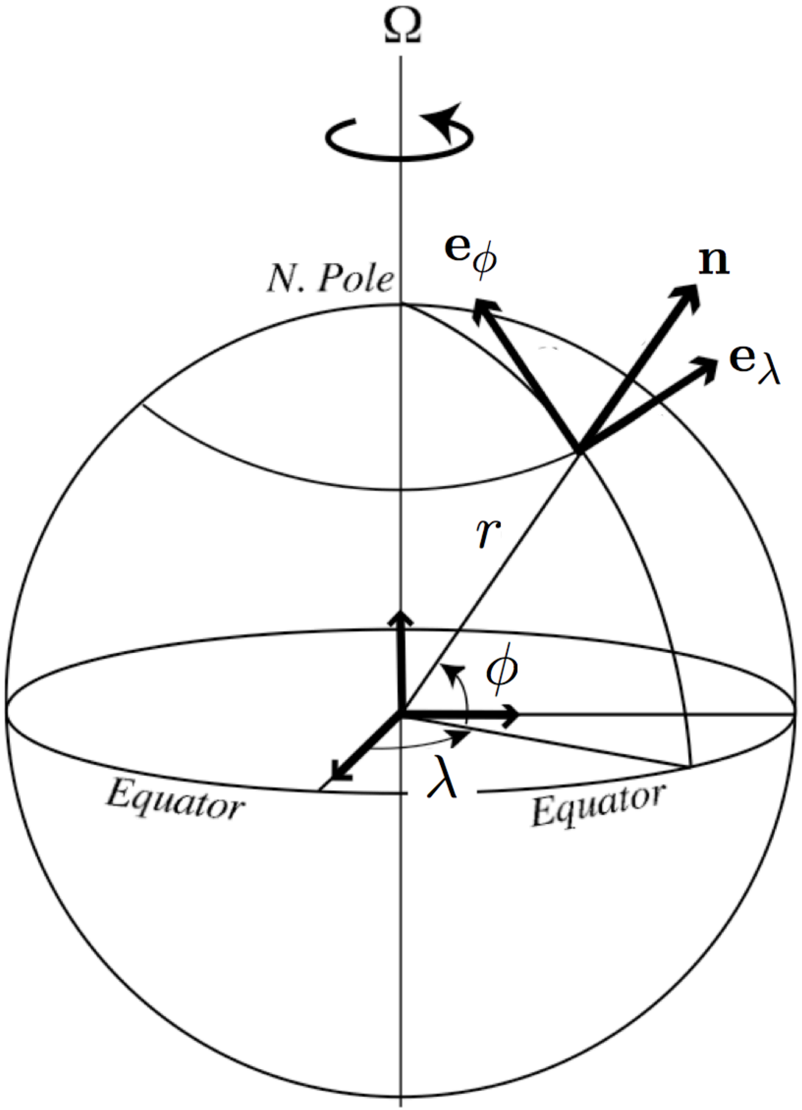
$$\frac{\mathrm{D}q}{\mathrm{D}t} = \frac{\partial q}{\partial t} + \sum_i \frac{\partial q}{\partial \xi^i} \frac{\mathrm{D}\xi^i}{\mathrm{D}t}$$

- the *contravariant* velocity components are:

$$u^i \equiv \frac{\mathrm{D}\xi^i}{\mathrm{D}t} \Longrightarrow \frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \sum_i u^i \frac{\partial}{\partial \xi^i}$$

Same advective form in curvilinear and Cartesiant coordinates. How about the flux form ? }

Flux form in curvilinear coordinates



Flux form in curvilinear coordinates

Consider as an example spherical coordinates (λ, φ, r)

- infinitesimal distance:

$$\begin{aligned} \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} &= r^2 \cos^2 \varphi \, \mathrm{d}\lambda^2 + r^2 \, \mathrm{d}\varphi^2 + \mathrm{d}r^2 \\ &= h_\lambda^2 \, \mathrm{d}\lambda^2 + h_\varphi^2 \, \mathrm{d}\varphi^2 + h_r^2 \, \mathrm{d}r^2 \end{aligned}$$

$$\text{where } \underbrace{h_\lambda = r\cos(\varphi) \qquad h_\varphi = r \qquad h_r = 1}_{\text{metric tensor}}$$

- **contravariant** vs "physical" :

$$\underbrace{\left(\dot\lambda, \dot\varphi, \dot r\right)}_{\text{contravariant}} \equiv \frac{\mathrm{D}}{\mathrm{D}t}(\lambda, \varphi, r) \qquad \underbrace{(u, v, w)}_{\text{physical}} = \left(h_\lambda \dot\lambda, h_\varphi \dot\varphi, h_r \dot r\right)$$

- volume element:

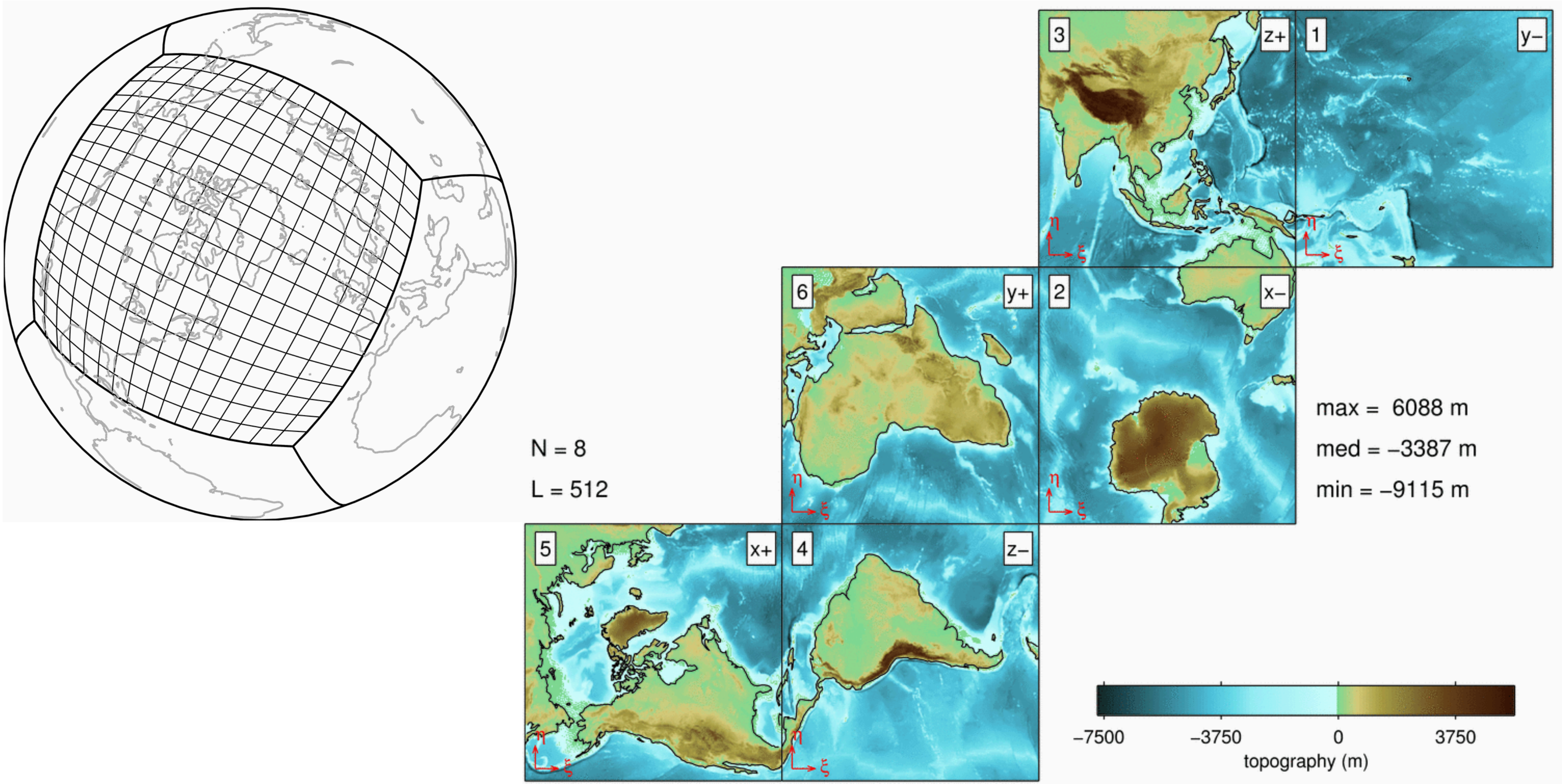
$$\mathrm{d}V = J \, \mathrm{d}^3\boldsymbol{x} \qquad \text{where} \qquad \underbrace{J = h_\lambda h_\varphi h_r}_{\text{Jacobian}}$$

- pseudo-density:

$$\mu = J\rho$$

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho\left(u\boldsymbol{e}_\lambda + v\boldsymbol{e}_\varphi + w\boldsymbol{e}_r\right) = 0 \quad \Longleftrightarrow \quad \frac{\partial \mu}{\partial t} + \frac{\partial}{\partial \lambda}\left(\mu\dot\lambda\right) + \frac{\partial}{\partial \varphi}(\mu\dot\varphi) + \frac{\partial}{\partial r}(\mu \dot r) = 0}$$

Flux form in curvilinear coordinates



Transport in curvilinear coordinates

$$\boxed{\begin{gathered} u^i \equiv \frac{\mathrm{D}\xi^i}{\mathrm{D}t} \qquad \partial_i \equiv \frac{\partial}{\partial \xi^i} \\ \frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial \xi^i} \\ \frac{\partial \mu}{\partial t} + \partial_i \left(\mu u^i \right) = 0 \end{gathered}}$$

Geometric approximations :

The shape of the Earth and its approximate representation

THEORIE
DE LA
FIGURE
DE LA TERRE,

Tirée des Principes de l'Hydrostatique.

Par M. CLAIRAUT, de l'Académie Royale des Sciences, & de la Société Royale de Londres.

J. L. De Traytorrens.

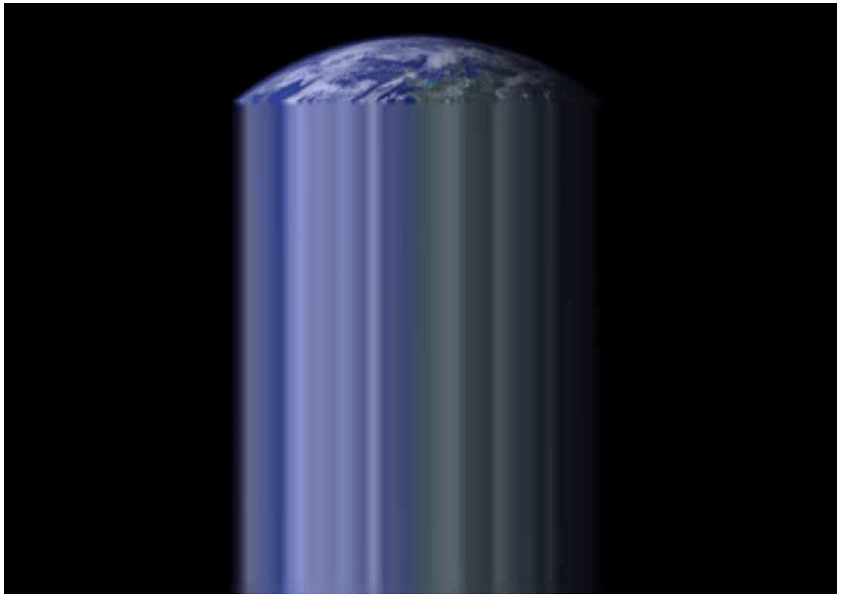
A PARIS,

Chez DURAND, Libraire, ruë Saint-Jacques, à Saint Landry & au Griffon.

MDCCXLIII.

AVEC APPROBATION ET PRIVILEGE DU ROY.

Motion in an inertial Cartesian frame



$$\frac{D\dot{\mathbf{x}}}{Dt} + \frac{1}{\rho}\nabla p + \nabla V = 0$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \dot{\mathbf{x}} = 0 \qquad \frac{Ds}{Dt} = 0$$

Specific entropy
= entropy per unit mass

Motion in a rotating Cartesian frame

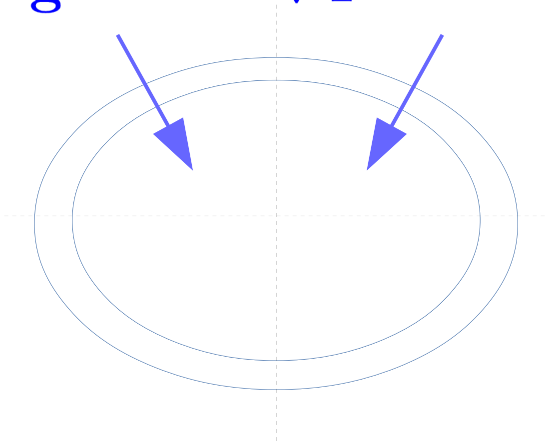
- Newton's fundamental principle of dynamics
- Forces : pressure and gravity
- Pseudo-forces : Coriolis and centrifugal

Planetary velocity

$$\mathbf{R} = \mathbf{\Omega} \times \mathbf{x}$$

Geopotential

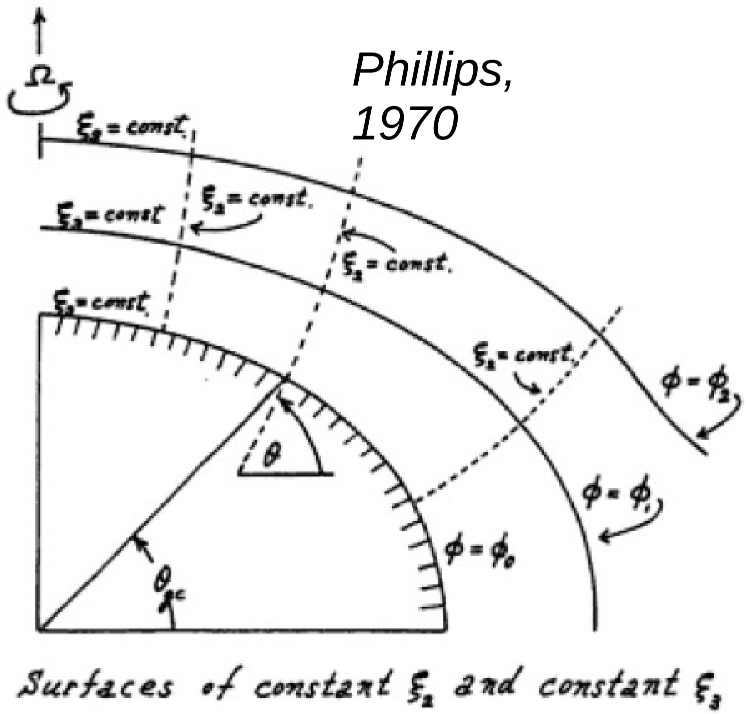
$$\Phi = V - \frac{1}{2}\mathbf{R}\cdot\mathbf{R}$$
$$\mathbf{g} = -\nabla\Phi$$



$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\ \mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\ \rho\dot{\mathbf{x}} = 0 \qquad \frac{Ds}{Dt} = 0$$

Dynamics in curvilinear coordinates



Phillips, 1970

$$\left(\xi^1, \xi^2, \xi^3\right) \mapsto \mathbf{x}\left(\xi^1, \xi^2, \xi^3\right)$$

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\ \mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

?

$$\mathbf{u} = \frac{D\mathbf{x}}{Dt}$$

$$\mathbf{u} \cdot \mathbf{u} = ?$$

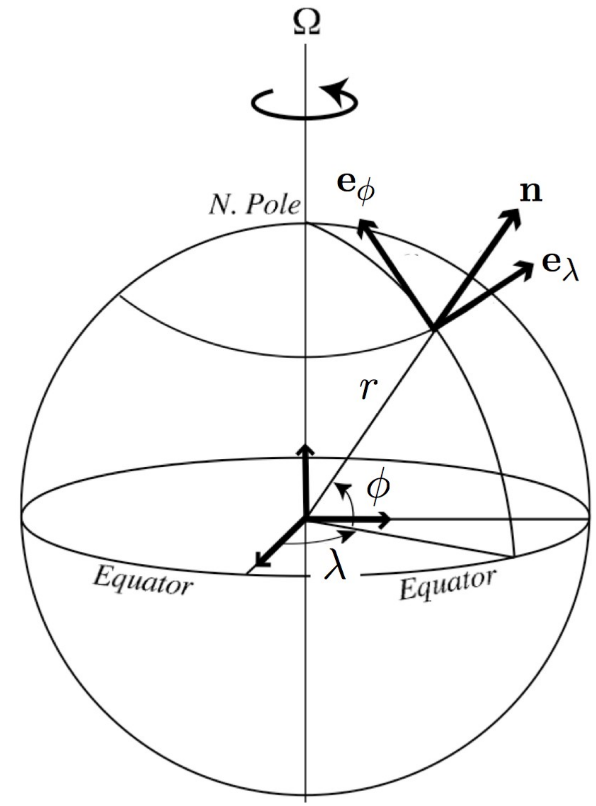
$$x = r\cos\phi\cos\lambda$$
$$y = r\cos\phi\sin\lambda$$
$$z = r\sin\phi$$

$$\left(\xi^1,\xi^2,\xi^3\right) \mapsto \mathbf{x}\left(\xi^1,\xi^2,\xi^3\right)$$

$$\mathbf{u} = \frac{D\mathbf{x}}{Dt}$$

$$\mathbf{u}\cdot\mathbf{u} = ?$$

$$\mathbf{e}_\lambda \equiv \frac{\partial \mathbf{x}}{\partial \lambda}$$
$$\mathbf{e}_\phi \equiv \frac{\partial \mathbf{x}}{\partial \phi}$$
$$\mathbf{e}_r \equiv \frac{\partial \mathbf{x}}{\partial r}$$



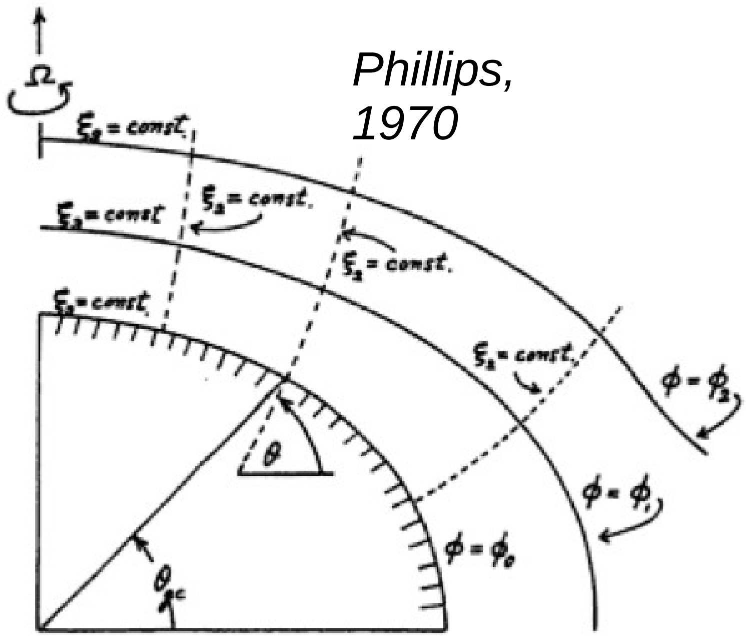
$$\begin{aligned}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} &= (\dot\lambda\mathbf{e}_\lambda + \dot\phi\mathbf{e}_\phi + \dot r\mathbf{e}_r)\cdot(\dot\lambda\mathbf{e}_\lambda + \dot\phi\mathbf{e}_\phi + \dot r\mathbf{e}_r)\\ &= \dot\lambda^2\mathbf{e}_\lambda\cdot\mathbf{e}_\lambda + \dot\phi^2\mathbf{e}_\phi\cdot\mathbf{e}_\phi + \dot r^2\mathbf{e}_r\cdot\mathbf{e}_r\\ &= G_{ij}u^iu^j = u^iu_i\end{aligned}$$

$$G_{\lambda\lambda} = r^2\cos^2\phi \qquad u_\lambda = r^2\cos^2\phi\dot\lambda \qquad R_\lambda = \Omega r^2\cos^2\phi$$
$$G_{\phi\phi} = r^2 \qquad u_\phi = r^2\dot\phi \qquad u_\phi = 0$$
$$G_{rr} = 1 \qquad u_r = \dot r \qquad u_r = 0$$

metric tensor G_{ij} — covariant velocity $u_i = G_{ij}u^j$ — covariant planetary velocity R_i

Dynamics in curvilinear coordinates

Phillips, 1970



Surfaces of constant ξ_2 and constant ξ_3

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\ \mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

?

$$\left(\xi^1, \xi^2, \xi^3\right) \mapsto \mathbf{x}\left(\xi^1, \xi^2, \xi^3\right)$$

$$\mathbf{u} = \frac{D\mathbf{x}}{Dt}$$
$$\mathbf{u} \cdot \mathbf{u} = ?$$

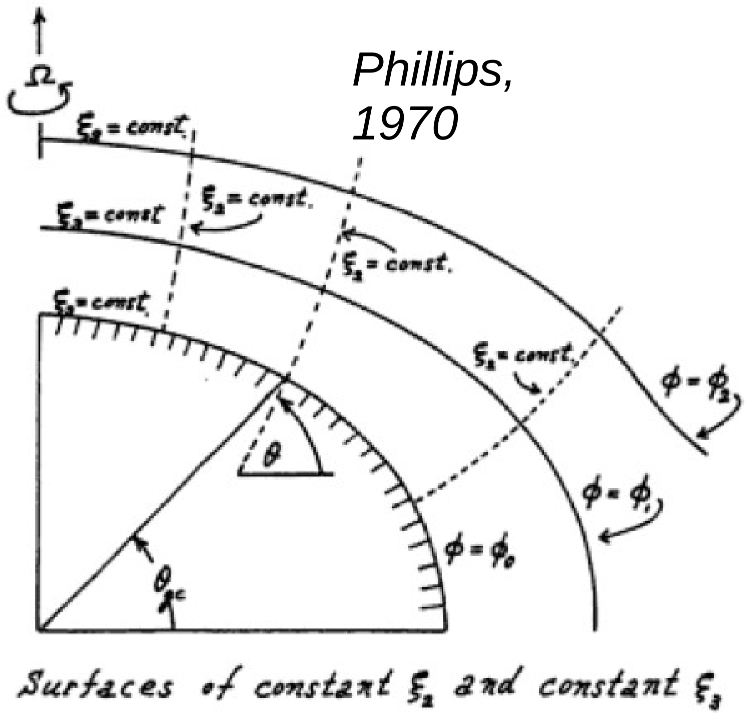
$$\begin{aligned} \mathbf{e}_i &= \partial_i \mathbf{x} \\ G_{ij}(\xi^k) &= \mathbf{e}_i \cdot \mathbf{e}_j \\ J &= (\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 \\ &= \sqrt{\det G_{ij}} \end{aligned}$$

metric

$$\begin{aligned} \mathbf{u} = \frac{D\mathbf{x}}{Dt} &= u^i \mathbf{e}_i \\ \mathbf{u} \cdot \mathbf{e}_i &= G_{ij} u^j = u_i \\ \mathbf{u} \cdot \mathbf{u} &= G_{ij} u^i u^j = u^i u_j \\ \mathbf{R} \cdot \mathbf{u} &= R_i u^i \end{aligned}$$

covariant components

Dynamics in curvilinear coordinates



Phillips, 1970

- *covariant* : same form in all coordinate systems
- derives from a *variational principle* : Hamilton's principle of least action
- *dynamically consistent* *(White & Bromley, 1995)* : conserves energy, angular momentum, potential vorticity for any choice of zonally-symmetric metric, planetary velocity, Jacobian

metric, planetary velocity, Jacobian can be approximated without jeopardizing *dynamical consistency*

various geometric approximations, each characterized by a certain choice of metric and planetary velocity

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\ \mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

$$\Updownarrow$$

$$\boxed{\begin{aligned} G_{ij}\frac{Du^j}{Dt} + \frac{1}{2}\left(\partial_j G_{ik} + \partial_k G_{ij} - \partial_i G_{jk}\right)u^j u^k \\ + \left[\partial_j R_i - \partial_i R_j\right]u^j + \frac{J}{\mu}\partial_i p + \partial_i \Phi &= 0 \end{aligned}}$$

(Tort & Dubos, 2014)

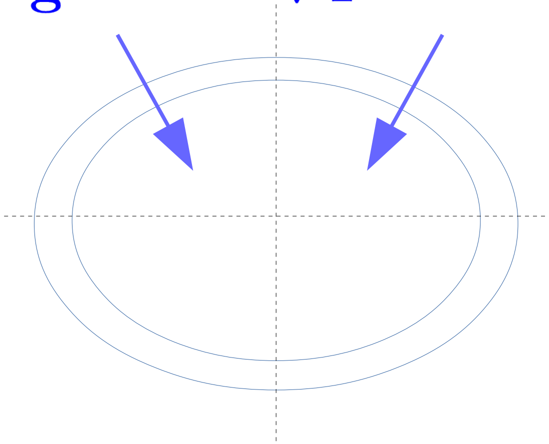
Spherical geoid approximation

- Ellipticity of geoids ~ centrifugal / gravitational ~ 1/300
- Spherical geoid approximation :
  pretend that the geoids (isosurfaces of geopotential) are spheres of radius $r(\Phi)$

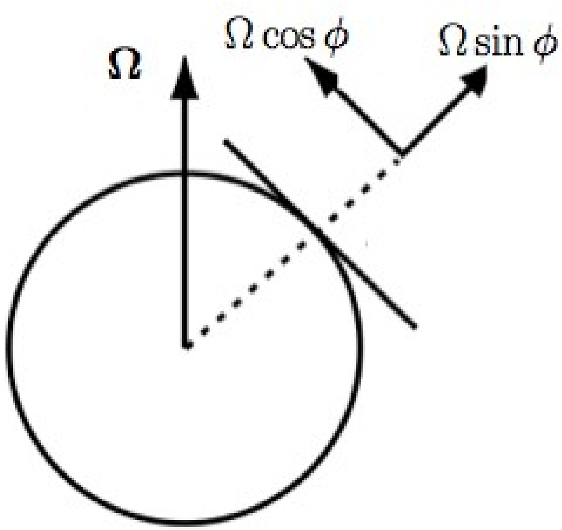
$$G_{ij}\mathrm{d}\xi^i\mathrm{d}\xi^j = r^2\cos^2\phi\mathrm{d}\lambda^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}r^2 \qquad R_j\mathrm{d}\xi^j = \Omega r^2\sin^2\phi\mathrm{d}\lambda$$

Geopotential

$$\begin{aligned}\Phi &= V - \frac{1}{2}\mathbf{R}\cdot\mathbf{R}\\ \mathbf{g} &= -\nabla\Phi\end{aligned}$$



Spherical geoid approximation



$$\begin{pmatrix} 0 \\ 2\Omega\cos\phi \\ 2\Omega\sin\phi \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2\Omega\sin\phi v + 2\Omega\cos\phi w \\ 2\Omega\sin\phi u \\ 2\Omega\cos\phi u \end{pmatrix} \begin{matrix} W \to E \\ S \to N \\ \end{matrix}$$

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right) v\sin\phi + 2\Omega\cos\phi\, w + \frac{uw}{r} + \frac{1}{\rho r\cos\phi}\partial_\lambda p = 0$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r\cos\phi}\right) v\sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\partial_\phi p = 0$$

$$\frac{Dw}{Dt} + (2\Omega\cos\phi)\, u - \frac{u^2+v^2}{r} + g + \frac{1}{\rho}\partial_r p = 0$$

Deep-atmosphere geometry

- atmospheric shallowness (radius a=6400 km >> r-a~50 km) suggests to let r=a, g(r)=g(a) and neglect blue terms : shallow-fluid approximation
- Small vertical velocities suggest to neglect red terms : traditional approximation
- Phillips (1966) : both approximations must be made together, otherwise angular momentum conservation is lost

	Earth	Titan
ratio	$\sim 1\%$	$\sim 25\%$
g variability	$\sim 2.8\%$	$\sim 53.4\%$

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right) v\sin\phi + 2\Omega\cos\phi\, w + \frac{uw}{r} + \frac{1}{\rho r\cos\phi}\partial_\lambda p = 0$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r\cos\phi}\right) v\sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\partial_\phi p = 0$$

$$\frac{Dw}{Dt} + (2\Omega\cos\phi)\, u - \frac{u^2+v^2}{r} + g + \frac{1}{\rho}\partial_r p = 0$$

Shallow-fluid and traditional approximations

Deep-atmosphere :

$$G_{ij}\mathrm{d}\xi^i\mathrm{d}\xi^j \equiv r^2\cos^2\phi\mathrm{d}\lambda^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}r^2 \qquad R_j\mathrm{d}\xi^j \equiv \Omega r^2\sin^2\phi\mathrm{d}\lambda$$

Atmospheric/oceanic shallowness (radius a=6400 km >> r-a~50 km) suggests to let r=a

Shallow-fluid : replace r by a in the metric

Traditional : replace r by a in the planetary velocity (Tort & Dubos, 2014)

$$G_{ij}\mathrm{d}\xi^i\mathrm{d}\xi^j \equiv a^2\cos^2\phi\mathrm{d}\lambda^2 + a^2\mathrm{d}\phi^2 + \mathrm{d}z^2$$

$$r = a + z$$
$$\Phi = gz$$

$$R_j\mathrm{d}\xi^j \equiv \Omega a^2\sin^2\phi\mathrm{d}\lambda \Longrightarrow \nabla\times R = f\mathbf{e}_z$$

f : Coriolis parameter

$$\frac{D\dot{\mathbf{x}}}{Dt} + f\mathbf{e}_z\times\dot{\mathbf{x}} + \nabla\Phi + \frac{1}{\rho}\nabla p = 0$$

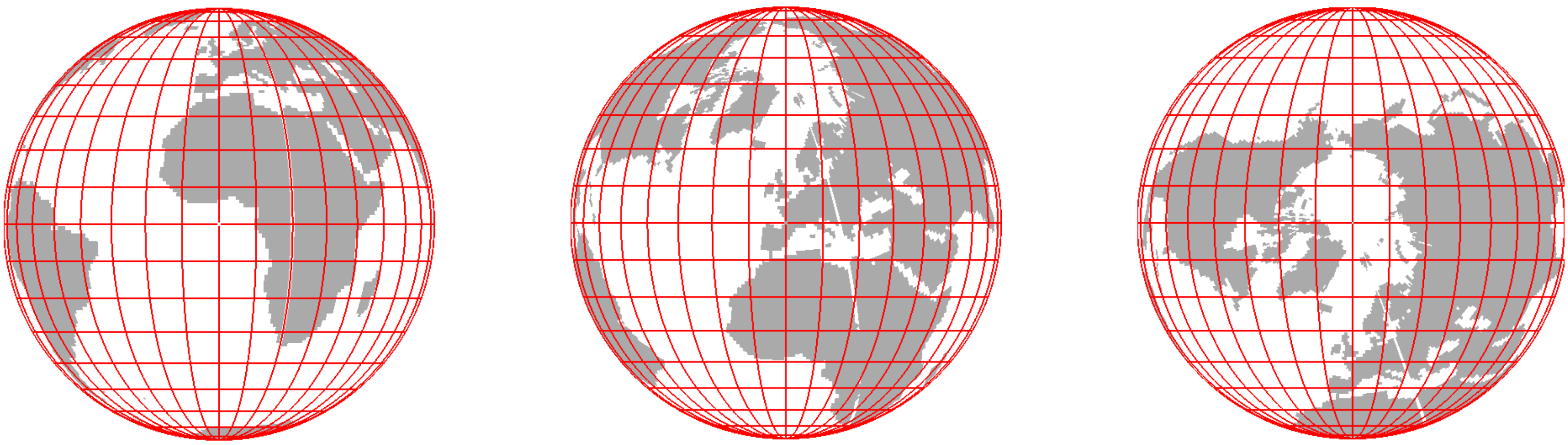
$$\boxed{\begin{aligned}
\frac{Du}{Dt} - \left(2\Omega + \frac{u}{a\cos\phi}\right)v\sin\phi + \frac{1}{\rho a\cos\phi}\partial_\lambda p &= 0\\
\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{a\cos\phi}\right)v\sin\phi + \frac{1}{\rho a}\partial_\phi p &= 0\\
\frac{Dw}{Dt} + (2\Omega\cos\phi)\,u + g + \frac{1}{\rho}\partial_z p &= 0
\end{aligned}}$$

$$\frac{D\dot{\mathbf{x}}}{Dt} + f\,\mathbf{e}_z \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

Traditional approximation : $\mathrm{curl}\ \mathbf{R} = f\,\mathbf{e}_z$

=> only the Coriolis parameter f matters for the dynamics

Why would the pole of the coordinate system be at the geographical pole ?
Let us put it at some arbitrary latitude … (Verkley, 1990)



$$f(\lambda,\phi) = 2\Omega\sin\phi$$

$$f(\lambda,\phi) = 2\Omega\cos\lambda\cos\phi$$

$$f(\lambda,\phi) = 2\Omega\left(\sin\phi\cos\phi_0 + \cos\lambda\cos\phi\sin\phi_0\right)$$

Tangent-plane approximations

Cartesian models with a simple expression of f

$$\frac{D\dot{\mathbf{x}}}{Dt} + f\,\mathbf{e}_z \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

Equatorial beta-plane

$$f = \beta y$$

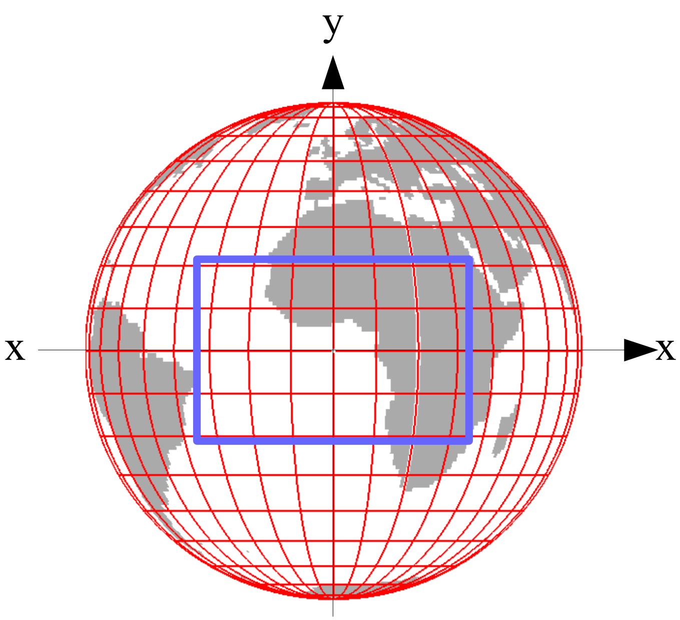
Beta-plane

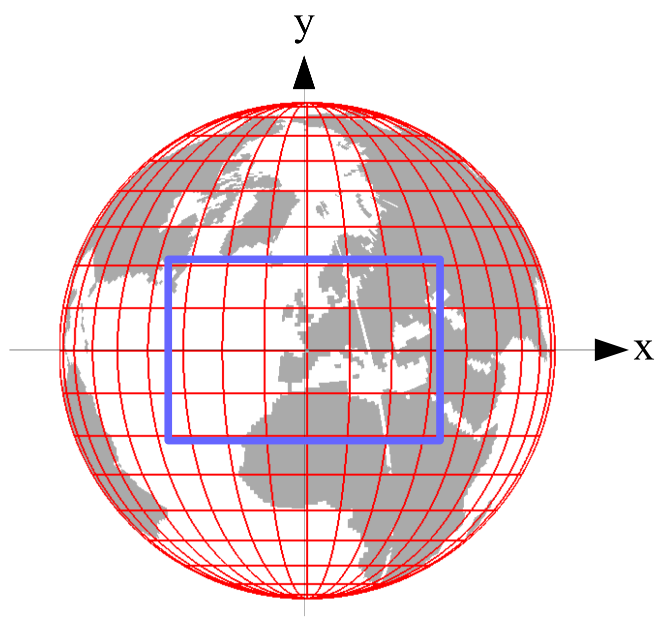
f-plane

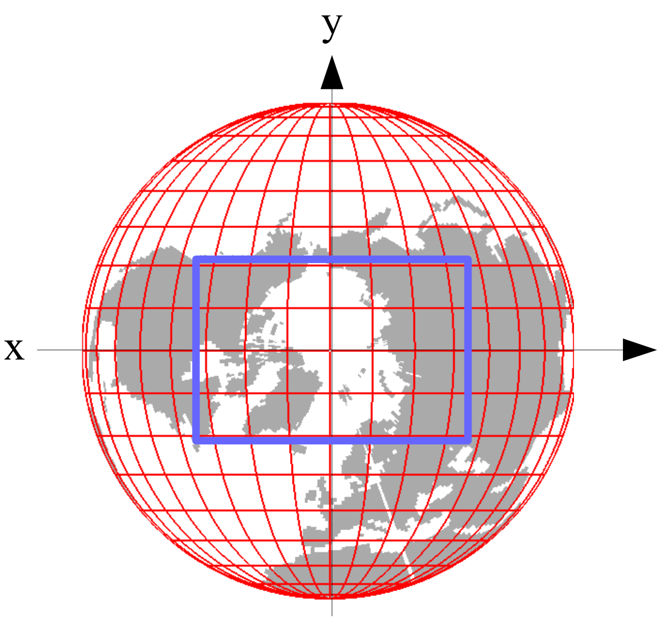
$$f = f_0 + \beta y$$

Delta-plane

$$f = 2\Omega\left(1 - \frac{x^2+y^2}{2a^2}\right)$$







$$f(\lambda,\phi) = 2\Omega\sin\phi$$

$$\phi = \frac{y}{a} \ll 1, \quad \lambda = \frac{x}{a} \ll 1$$

$$f(\lambda,\phi) = 2\Omega\cos\lambda\cos\phi$$

$$f(\lambda,\phi) = 2\Omega\left(\sin\phi\cos\phi_0 + \cos\lambda\cos\phi\sin\phi_0\right)$$

Wrap-up

Inertial Cartesian ⇔ Rotating Cartesian ⇔ Rotating Curvilinear ⇒ Rotating Curvilinear

Gravity

Planetary velocity
Geopotential
Coriolis

Contravariant
Covariant

Exact geometry

↓

spherical geoid

↓

shallow-fluid
traditional

↓

beta-plane

↓

f-plane

Fully compressible fluid

↓ ↓

Boussinesq/anelastic *Hydrostatic*

↓ ↓

Hydrostatic Boussinesq

- Geometry = metric tensor + planetary velocity
- Affects the equations of dynamics (momentum)
- But **transport equations** are the same in **any coordinate system**
- Flux-form <=> advective form

References

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